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On Banach Fixed Point Theorem

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Abstract

In this work, we study the Banach Fixed Point Theorem with a view to having an insight into the implication of varied contraction constants for a family of contraction maps. We prove the existence of fixed points of a family of contraction operators defined on a complete metric space. We also prove that the sequence of fixed points of a family of contraction operators defined on a complete metric characterised by a family of contraction constants converges. Finally, using examples we demonstrate that the fixed point (which it converges to) is well behaved and represent the fixed points of a family of contraction operators for given initial value.

1.0. Introduction

A lot of research effort and interest has followed the Banach Fixed Point Theorem since 1922, opening a research area in functional analysis called fixed point theory. This basic theorem has been extended, generalised and enriched in many directions. It has been applied to solve problems in nonlinear analysis, differential equations, dynamical systems, optimization etc. [1, 2, 4, 5] bear records of some research efforts in fixed point theory.

The theorem asserts that a space is a fixed-point space relative to a family maps defined on it. Specifically, it asserts that any complete metric space is a fixed-point space relative to the family of contraction self-maps on it.

In 1968, Nadler showed that for a locally compact complete metric space A', a sequence x_n (respectively x) of fixed points of $F_n : A' \to A'$ (respectively of $F : A' \to A'$) converges to x.

In this work, we study the family of contractive self-maps on a complete metric space and furthermore show that a sequence of this maps need not converge for the sequence of fixed points x_n to converge to x. We prove that with some conditions imposed on the sequence of contraction constants for the maps, x_n converges to x.

2.0. Preliminaries

We begin with some basic definitions, concepts and results in metric space and metric fixed point theory needed in the sequel.

Definition 2.1[3]
Let X be any nonemtpty set. Then the function $d: X \times X \rightarrow \mathbb{R}$ is said to be a metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

1.1a: $d(x, y) \geq 0$

1.1b: $d(x, y) = 0$ iff $x = y$

1.1c: $d(x, y) = d(y, x)$

1.1d: $d(x, y) \leq d(x, z) + d(z, y)$

Thus the pair (X, d) is called a metric space.

Definition 2.2[6]
The sequence x_n in (X, d) is convergent with limit x, if as $n \rightarrow \infty$, $d(x_n, x) \rightarrow 0$.

Definition 2.3[6]
The sequence x_n in (X, d) is said to be a Cauchy sequence if for any $m > p$ we have that $d\left(x_p, x_m\right) \rightarrow 0$ as $n \rightarrow \infty$

Definition 2.4[3]
A metric space (X, d) is said to be complete if every Cauchy sequence $x_n \in X$ converges.
Our space of interest shall be a complete metric space.

Definition 2.5[2]:
Let (X, d) be a metric space. A map $T: X \rightarrow X$ is said to be a *contraction* if for each $x, y \in X$, there exists a constant $k \in [0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (1)$$

Definition 2.6[5]:

Let (X, d) be a metric space (in general any space) and T any map on X or a subset of X into X. Then a point $x \in X$ is called a fixed point of T if $Tx = x$

Definition 2.7[2]:

Let X be any set and $T: X \rightarrow X$ be a self map on X. The for a given $x \in X$ $T^n x$ is said to be the nth iterate of x under T and the set $\{T^n x: n = 0,1,2,.,.,\}$ is the orbit of x under T where $T^n x$ is defined inductively as $T^0 x = x$ and $T^{n+1} x = T(T^n x)$. It is popularly called the Picard iteration.

Definition 2.8[5]:

Let (X, d) be a metric space (in general any space) and T any map on X or a subset of X into X. Then the space X is called a

2.8.1: fixed point space if every map $T: X \rightarrow X$ has a fixed point.

2.8.2: fixed point space relative to a family of maps M if each $f \in M$ has a fixed point.

Thus, (X, d) (in general any space) may not be a fixed point space. But with some well-defined property on the map, X(in general any space) we can have fixed point.

An example of 2.8.2 is the assertion of the most popular theorem in fixed point theory: Banach Fixed Point Theorem

Theorem 2.9 (Banach) [3]

Consider a metric space (X, d), where $X \neq 0$. Suppose that X is complete and let $T: X \to X$ be a contraction on X. Then T has precisely one fixed point.

Proof:**[3]**

In this work, we refer to k as the contraction constant.

Corollary 2.10 (Iteration, error bounds) [3]
Under the conditions of the theorem above, an iterative sequence with arbitrary $x_0 \in X$ converges to the unique fixed point x of T.
The prior error estimate is given as
$d(x_m, x) \leq \frac{k^m}{1-k} d(x_0, x_1)$.. ....(2)
and the posterior estimate is given as

$d(x_m, x) \leq \frac{k}{1-k} d(x_{m-1}, x_m)$.. ....(3)

3. 0. Main Work

We introduce and give some properties and examples of single-valued functions defined on partially ordered set. these functions will be useful in the study of the behaviour of contraction constants and their influence on a sequence of contractive functions we shall define later.
Definition 3.1: Let $\sigma: E \to [0,1)$ be a single-valued map on a partially ordered set E $(E, \leq)$ into $[0,1)$. We call σ

3.1.1: an **in** function if $\sigma(e_i) \leq \sigma(e_{i+1}) \Rightarrow e_i \leq e_{i+1} \quad \forall e_i \in E$

3.1.2: a **de** function if $\sigma(e_i) \geq \sigma(e_{i+1}) \Rightarrow e_i \geq e_{i+1} \quad \forall e_i \in E$

3.1.3: a **constant** function if $\sigma(e) = c \quad \forall e \in E$

3.1.4: an in-**de** function if $e_i \geq e_{i+1} \Rightarrow \sigma(e_i) \leq \sigma(e_{i+1}) \quad \forall e_i \in E$

3.1.5: an **de-in** function if $e_i \leq e_{i+1} \Rightarrow \sigma(e_i) \geq \sigma(e_{i+1}) \quad \forall e_i \in E$

Example 3.2:
Let $\sigma: \mathbb{R}^+ \to [0,1)$ be a single-valued map defined by $\sigma(k) = \frac{1}{k}$ $, \forall k \in \mathbb{R}$ and $\sigma(0) = 0$

Then σ satisfy 3.1.4 and 3.1.5

Example 3.3:
Let $\sigma: \mathbb{R}^+ \to [0,1)$ be a map defined by $\sigma(k) = \frac{2^k}{1+2^k}$. Clearly, $\sigma(k) \in [0,1)$ for all $k \geq 0$.

Then σ satisfy 3.1.1

Example. 3.4:
Let $\sigma: \mathbb{R}^{+} \to [0,1)$ be a map defined by $\sigma(k) = \frac{k}{1+2^k}$. Clearly, $\sigma(k) \in [0,1) \; \forall \; k \geq 0$. Then σ satisfy 3.1.5

Definition 3.5:
Let $\sigma: E \to [0,1)$ and $f_i: X \to X$, a contraction operator with contractive constant $\sigma(e_i)$ for each i.

Define a family $\mathcal{F}$ of contraction operators:

$\mathcal{F} := \{f_i: \; \forall i \in N\}$

Example 3.6
Consider the maps $\sigma: \mathbb{R} \to [0,1)$ and $f: X \to \mathbb{R}, X \subset \mathbb{R}$ such that $\sigma(r_i) = \frac{1}{r_i}$ and $f_i(x) = \sigma(r_i).x + b$, b, any positive constant in $\mathbb{R}$. Clearly $f_i \in \mathcal{F}$ as each $\sigma(r_i) \in [0,1)$.
Note that $\sigma(r_i) \geq \sigma(r_{i+1})$ as $r_i \leq r_{i+1}$ for all $i \in N$.

Example 3.7
Consider the maps $\sigma: \mathbb{R} \to [0,1)$ and $T: \mathbb{R} \to (1, \infty)$ such that $\sigma(r_i) = \frac{1}{r_i}$ and $f_i(x) = x^{\sigma(r_i)} - x - b$, b, any constant in $\mathbb{R}$. Clearly $f_i \in \mathcal{F}$ as each $\sigma(r_i) \in [0,1)$, for all $i \in N$.

Example 3.8
Consider the maps $\sigma: \mathbb{R} \to [0,1)$ and $T: \mathbb{R} \to (1, \infty)$ such that $\sigma(k_i) = \frac{2^{k_i}}{1+2^{k_i}}$ $k_i \geq 0$ and $f_i(x) = \sigma(k_i).x + b$, b, any constant in $\mathbb{R}$. Clearly $f_i \in \mathcal{F}$ as each $\sigma(k_i) \in [0,1)$, for all $i \in N$.

Next, we introduce a family of contraction constant on a metric space.
Definition 3.8:
Let (X, d) metric space. A family $\mathcal{F}$ of contraction operators $T_i: X \to X$ with contractive constant $\sigma(e_i)$ for each $i \in N$ is said to be a $\boldsymbol{\sigma - function\ contraction}$ if for each $x, y \in X$, there exists a function $\sigma: E \to [0,1)$ with $\sigma(0) = 0$ such that for any $e_i \in E$

$$d(T_i x, T_i y) \leq \sigma(e_i).d(x, y) \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (4)$$

The following are the results of his study.

Proposition 3.9:
Let (X, d) be a complete metric space. Suppose $T_i: X \to X$ is a $\sigma - function\ \ contraction$ on X for each i. Then for each $e_i \in E$ T_i has a unique fixed point.

Proof:
Putting $\sigma(e_i) = c$ in theorem 2.9 above as each $\sigma(e_i) \in [0,1)$, there exist a unique fixed point of T_i , $say\ x_i^*$, such that $T_i(x_i^*) = x_i^*$. So for each $e_i \in E$ there exist a unique fixed point x_i^* and the proof is complete**.**

Proposition 3.10:
Let (X, d) be a complete metric space. Suppose the family $\mathcal{F}$=$\{T_i: T_i$ a $\sigma - function\ \ contraction$ on $X)$. Then for any sequence of points $\{e_i\}_{i=1}^{n}$ of E , T has a sequence of unique fixed points $\{x_i^*\}_{i=1}^{n}$.

Proof:
By proposition 3.9, there exist a unique fixed point of T_i , $say\ x_i^*$ for each $\sigma(e_i) \in [0,1)$, such that $T_i(x_i^*) = x_i^*$ .So for any sequence of points $\{e_i\}_{i=1}^n$ of E $e_i \in E$ there exist a unique sequence of unique fixed points $\{x_i^*\}_{i=1}^n$.

Proposition 3.11:
Suppose (X,d) satisfies proposition 3.1, then the rate of convergence is given as $d(x_m, x_i^*) \leq \sigma(e_i)^m d(x_0, x_i^*)$ where $e_i \in E$, $m = 1,2,.,.,.$

Proof:
Let $T_i(x_0)= x_1$, $T_i{}^2(x_1)=T_i(T_i(x_0))$, $T_i(x_2)= T_i\ (T_i{}^2(x_1))= T_i\ (T_i(T_i(x_0)))$, . , . , . , $T_i{}^m(x_{m-1}) = T_i\ (T_i{}^{m-1}(x_{m-2}))= . . . . =T_i\ (T_i \dots (T_i(x_0)))$ be the picard iteration and

$$x_{m+1} = T_i(x_m) = T_i x_m \qquad (5)$$

be the sequence of successive approximations. Then for any i with fixed point x_i^* of T_i, then $d(x_m, x_i^*) = d(T_i x_{m-1}, x_i^*) \leq \sigma(e_i)\ d(x_{m-1}, x_i^*) = \sigma(e_i)\ d(T_i x_{m-2}, x_i^*) \leq \sigma(e_i)^2\ d(x_{m-2}, x_i^*) \dots$

$=\sigma(e_i)^{m-1}\ d(T_i x_0, x_i^*) \leq \sigma(e_i)^m\ d(x_0, x_i^*)$, since each $\sigma(e_i) \in [0,1)$. The proof is complete.

Proposition 3.12:

Suppose (X,d) satisfy definition 3.1 and with the property above then the priori error estimate is given as
$d(x_m , x_i^*) \leq \frac{\sigma(e_i)^m}{1-\sigma(e_i)}\ d(x_0, x_1)$.......................(6)
and the posterior estimate is given as

$d(x_m , x_i^*) \leq \frac{\sigma(e_i)}{1-\sigma(e_i)}\ d(x_{m-1}, x_m)$................(7)
where $e_i \in E$, $m = 1,2,...$

Proof:

Recall $that d(T_i x, T_i y) \leq \sigma(e_i). d(x,y)$. Now $x_{m+1} = T_i(x_m) = T_i x_m$. Then $x_2 = T_i x_1$, $x_1 = T_i x_0$

which implies $that\ d(x_2, x_1) = d(T_i x_1, T_i x_0) \leq \sigma(e_i). d(x_1, x_0)$. Inductively, we have $d(x_{m+1}, x_m) \leq \sigma(e_i)^m. d(x_1, x_0)$, where $e_i \in E$, $m = 1,2,..$ Let $q \in \mathbb{N}$, $q > 0$, then we have $d\big(x_{m+q}, x_m\big) \leq \sum_{j=m}^{m+q-1} d(\ x_{j+1}, x_j) \leq \sum_{j=m}^{m+q-1} \sigma(e_i)^j\ d(x_1, x_0) \leq \frac{\sigma(e_i)^j}{1-\sigma(e_i)}\ d(x_1, x_0)$. $As\ q \to \infty$, $x_{m+q} \to x_i^*\ for\ each\ e_i \in E$. So $d(x_i^*\ , x_m) = \lim_{q \to \infty} d\big(x_{m+q}, x_m\big) \leq \frac{\sigma(e_i)^m}{1-\sigma(e_i)}\ d(x_1, x_0)$. Therefore $d(x_m, x_i^*) \leq \frac{\sigma(e_i)^m}{1-\sigma(e_i)}\ d(x_1, x_0)$ and the proof of equation 6 is complete.

Next, we prove equation 7 by induction. i.e $d(x_{m+1}, x_m) \leq \sigma(e_i) d(x_m , x_{m-1})$ inductively gives $d\big(x_{m+j} , x_{m+j-1}\big) \leq \sigma(e_i)^j d(x_m , x_{m-1})$. This gives $d\big(x_{m+q} , x_m\big) \leq (\sigma(e_i) + \sigma(e_i)^2 + \cdots + \sigma(e_i)^q) d(x_m , x_{m-1})$. But $\frac{\sigma(e_i)}{1-\sigma(e_i)} = (\sigma(e_i) + \sigma(e_i)^2 + \cdots + \sigma(e_i)^q$. So $d\big(x_{m+q} , x_m\big) \leq \frac{\sigma(e_i)}{1-\sigma(e_i)} d(x_m , x_{m-1})$, which is equation 7 and the proof of proposition 3.12 is complete.

Proposition 3.13:
If for proposition 3.11, σ satisfies definition 3.1.3, then proposition 3.11 becomes proposition 3.13 i.e for $\sigma(e_i) = c \Rightarrow \forall i$ then $d(T_i x, T_i y) \le \sigma(e_i).d(x,y)$ becomes $d(T_i x, T_i y) \le c.d(x,y)$.

Proof: Trivial

So, the famous Banach fixed point theorem satisfies proposition 3.1, hence proposition 3.1 is more general.

Proposition 3.14:

Let (X,d) be a complete metric space. Suppose $T_i: X \to X$ be a $\sigma - function\ \ contraction$ on X and σ satisfies proposition 3.4, then for an initial $x_0 \in X$, the priori estimates decreases as σ decreases resulting to faster rate of convergence.

Proof:

Now for an initial $x_0 \in X$ and $e_i \in [0,1)$, there exist fixed points x_i^* for T_i by proposition 3.4. Also the prori estimate

$$d(x_m, x_i^*) \le \frac{\sigma(e_i)^m}{1-\sigma(e_i)} d(x_0, x_1) \text{.............................}(8)$$

By definition1.8, let $e_1, e_2, ., ., e_{i,e_{i+1}}, .$ $\in E$ such that $\sigma(e_1), \sigma(e_2), ., ., \sigma(e_i), \sigma(e_{i+1}), .$ $\forall i$ with fixed Then equation 3.5 gives fixed points $x_1^*, x_2^*, ., ., x_i^*, x_{i+1}^*, .,$ $\forall i$ having priori estimates $d(x_m, x_1^*) \le \frac{\sigma(e_1)^m}{1-\sigma(e_1)} d(x_0, x_1)$ for $\sigma(e_1)$, $d(x_m, x_2^*) \le \frac{\sigma(e_2)^m}{1-\sigma(e_2)} d(x_0, x_1)$ for $\sigma(e_2)$, $d(x_m, x_i^*) \le \frac{\sigma(e_i)^m}{1-\sigma(e_i)} d(x_0, x_1)$ for $\sigma(e_i)$ and

$d(x_m, x_{i+1}^*) \le \frac{\sigma(e_{i+1})^m}{1-\sigma(e_{i+1}))} d(x_0, x_1)$ for $\sigma(e_{i+1})$, $\forall i$........ (9)

Since σ satisfies definition 3.1.2 i.e $\sigma(e_i) \ge \sigma(e_{i+1})$, then m$\frac{\sigma(e_i)^m}{1-\sigma(e_i)} \ge \frac{\sigma(e_{i+1})^m}{1-\sigma(e_{i+1})}$.

Let $0 \le k < 1$ such that
$\sigma(e_i) = \sigma(e_{i+1}) + k$, so $\sigma(e_{i+1}) + k \in [0,1)$, then
$\frac{\sigma(e_i)^m}{1-\sigma(e_i)} = \frac{(\sigma(e_i)+k)^m}{1-(\sigma(e_i)+k))}$...(10)

So error estimates increase as σ increases. Now since the error estimate shows that for an initial x_0, the approximation error of the m^{th} iterate is determined by the contraction coefficient σ. Now putting equation 10 into equation 9 gives $(x_m, x_i^*) \le \frac{\sigma(e_i)^m}{1-\sigma(e_i)} d(x_0, x_1) = \frac{(\sigma(e_{i+1})+k)^m}{1-(\sigma(e_{i+1})+k))} \ d(x_0, x_1)$.

$Let\ n \in N$ such that $m - n > 0$ so that $\frac{(\sigma(e_{i+1})+k)^m}{1-(\sigma(e_{i+1})+k))} = \frac{(\sigma(e_i))^n}{1-(\sigma(e_i))}$. Then $d(x_m, x_i^*) \le \frac{(\sigma(e_i))^n}{1-(\sigma(e_i))} d(x_0, x_1)$, which shows that as the contraction function σ decreases, the number of iterates decreases i.e T converges to its fixed point faster.
Alternatively, we can prove T converges faster as σ decreases. Recall the rate of convergence is given as $d(x_m, x_i^*) \le \sigma(e_i)^m d(x_0, x_i^*)$, where $e_i \in E, m = 1,2, ., .$ Now for a decreasing sequence $\sigma(e_i) \ge \sigma(e_{i+1})$ then $d(x_m, x_{i+1}^*) \le \sigma(e_{i+1})^m d(x_0, x_{i+1}^*) \le \sigma(e_i)^m d(x_0, x_i^*)$. So $d(x_m, x_{i+1}^*) \le d(x_m, x_i^*)$ and the proof is complete.

Next, we prove the uniqueness of each fixed point of a family of contractive operators in a complete metric space. We also show that given certain conditions the sequence of fixed points converge.

Proposition 3.15:
Let (X,d) be a complete metric space. Suppose $T_i: X \rightarrow X$ is a $\sigma - function\ \ contraction$ on X. Then for each $e_i \in E$, T_i has one fixed point x_i^*. Furthermore, let $x_1^*, x_2^*, . , . , x_i^*$ be a sequence of fixed points of T for each $\sigma(e_i) \in [0,1)$ having the same initial guess $x_0 \in X$, then $x_1^*, x_2^*, . , . , x_i^*$ converges to a fixed point in the sequence say x^{**} as $\sigma(e_i) \rightarrow 0$ and $x_i^* \geq x^{**}$ if σ is as defined in 3.1.1 above.

Proof:
We need to show that $d(x^{**}, x_i^*) \rightarrow 0$ as $\sigma(e_i) \rightarrow 0$. For if m is fixed so that $x_m = x^{**}$, say. In proposition 1.11 above, $d(x_m, x_i^*) \geq d(x_m, x_{i+1}^*)$ as $\sigma(e_i) \geq \sigma(e_{i+1})$. This implies that $d(x^{**}, x_i^*) \geq d(x^{**}, x_{i+1}^*)$ as $\sigma(e_i) \rightarrow 0$.i.e $d(x^{**}, x_i^*) \rightarrow 0$ as $\sigma(e_i) \rightarrow 0$. So $\lim_{\sigma(e_i) \rightarrow 0} x_i^* = x^{**}$ and the proof is complete.

Definition 2.21:
Let (X,d) be a complete metric space. Suppose $T_i: X \rightarrow X$ is a $\sigma - function\ \ contraction$ on X and $\mathcal{F}$={T_i: T_i a contraction map} such that the sequence $x_1^*, x_2^*, . , . , x_i^*$ of fixed points of T_i for each $\sigma(e_i) \in [0,1)$ having the same initial guess $x_0 \in X$, converges to a fixed point in the sequence say x^{**}. We call x^{**} the *Limit fixed point (**LFP**) of* $\mathcal{F}$ for a $\sigma - function$.

Definition 3.4:
Let (X,d) be a complete metric space. Suppose $T_i: X \rightarrow X$ is a $\sigma - function\ \ contraction$ on X with x^{**} *Limit fixed point of* T for a $\sigma - function$. Then we call the $\sigma(e) \in [0,1)$ such that $T(x^{**}) = x^{**}$ the $\sigma^{**} - constant$ of $\mathcal{F}$.

4.0. Some Applications

Fixed point methods have proved to be efficient tools in solving linear and non-linear equations. Such equations are restated as fixed point problems and using an appropriate iteration scheme, so that for any initial point say x_0 in the space, the sequence of successive approximations converges to a point say x^*in the space with the property $T(x^*) = x^*$. Such fixed point is the solution to the restated equation.

Here we consider some of such problems: sequences of equations restated as fixed point problems with maps that are contractive. We show that the sequence of maps is $\sigma - function\ \ contraction$ on X. The results of computation of fixed points and limit fixed point using he Picard iteration scheme is also provided .

Example 4.1:
Let $x - \frac{x}{e} - b = 0$ where $\sigma: E \in \mathbb{R} \setminus \{0,1\} \rightarrow (1, \infty)$ such that $\sigma(e) = \frac{1}{e}$, $E \in \mathbb{R}$, b any constant and d the usual metric on $\mathbb{R}$, $t$hen $x = \frac{x}{e} + b$.

We show that the self map $T: X \rightarrow X$ such that $T(x) = \frac{x}{e} + b$ is a $\sigma - function\ \ contraction$ on X. Clearly, $T(x) \in \mathcal{F}$. Now, let $x, y \in X$, then $d(Tx, Ty) = |\sigma(e).x + b - (\sigma(e).y + b)| = \left|\frac{x}{e} + \text{b} - \frac{y}{e} - \text{b}\right|$

$\leq \left|\frac{x}{e} - \frac{y}{e}\right| = \frac{1}{e}|x - y| = \frac{1}{e}.d(x,y) = \sigma(e).d(x,y)$. So $d(Tx, Ty) \leq \sigma(e).\ d(x,y)$. Thus T is a $\sigma - function$ *contraction* on X .

Example 4.2:
Consider the equation $x^{\sigma(r)} - x - 1 = 0$ with $f: E \rightarrow (1, \infty)\ X, E \in \mathbb{R}$.

So $x^{f(e)} - x - 1 = 0 \implies x^{f(e)} = x + 1 \implies x = (x+1)^{\frac{1}{f(e)}}$. We have that the self map $T: X \longrightarrow X$ such that $T(x) = (x+1)^{\frac{1}{f(e)}}$ is a $\sigma - function\ contraction$ with $\sigma(e) = \frac{1}{f(e)} < 1$. Clearly, $T(x) \in \mathcal{F}$
With $b = 1\ and\ \sigma(e) = \frac{1}{f(e)}$ in example 1.5.2 . Now $d(Tx, Ty) = \left|(x+1)^{\frac{1}{f(e)}} - (y+1)^{\frac{1}{f(e)}}\right|$
$\leq \frac{1}{f(e)}|x + 1 - (y+1)| = \frac{1}{f(e)}|x - y| = \frac{1}{e}.d(x,y) = \sigma(e).d(x,y)$. So $d(Tx,Ty) = \sigma(e).\ d(x,y)$. So T is a $\sigma - function\ \ contraction$ on X .

Example 4.3:
Consider the equation of the form $x - \frac{2^k}{1+2^k}x - 3 = 0$, where $\sigma: K \rightarrow (1,\infty)$ $X \in \mathbb{R}, K \in \mathbb{R}$. Then $x = \frac{2^k}{1+2^k}x + 3$.
We show that the self map $T: X \to X$ such that $T(x) = \frac{2^k}{1+2^k}x + 3$ is a $\sigma - function\ \ contraction$ on X . Clearly, $T(x) \in \mathcal{F}$ with $b = 3\ \ and\ \ \sigma(k) = \frac{2^k}{1+2^k}$ in example 1.5.3. Now $d(Tx,Ty) = \left|\frac{2^k}{1+2^k}x + 3 - \left(\frac{2^k}{1+2^k}y + 3\right)\right| = \left|\frac{2^k}{1+2^k}.x - \frac{2^k}{1+2^k}.y\right| = \frac{2^k}{1+2^k}|x - y| = \sigma(k)\ d(x,y)$. Then $d(Tx,Ty) = \sigma(k)\ d(x,y)$

for each $k \geq 0$. So T is a $\sigma - function\ \ contraction$ on X .

5.0. Computations
Using the Picard iteration scheme, we compute the fixed points and the Limit Fixed Point for each sequence of self-maps above. Our computation shows that for an initial point

1. the fixed points converge as the contraction constants converge
2. the number of iterations converge as the contraction constants converge

5.1: Here we show the behaviour of the equation of Example 4.1. above with $b = 3$.
$x = \frac{x}{e} + 3$, $x_0 = 0,$ $x_i^{**} = 3.0151$

	$\sigma(e)$	$\boldsymbol{Tx_m}$	Fixed point	m
$x = \frac{x}{2} + 3$	$\frac{1}{2}$	16.0000 3.0000 4.5000 5.2500 5.6250 5.8125 5.9063 5.9531 5.9766 5.9883 5.9941 5.9971 5.9985 5.9993 5.9996 5.9998 5.9999	5.9999	17
$x = \frac{x}{6} + 3$	$\frac{1}{6}$	9.0000 3.0000 3.5000 3.5833 3.5972 3.5995 3.5999 3.6000 3.6000	3.6000	8
$x = \frac{x}{8} + 3$	$\frac{1}{8}$	8.0000 3.0000 3.3750 3.4219 3.4277 3.4285 3.4286	3.4286	7
$x = \frac{x}{10} + 3$	$\frac{1}{10}$	7.0000 3.0000 3.3000 3.3300 3.3330 3.3333	3.3333	6
$x = \frac{x}{15} + 3$	$\frac{1}{15}$	6.0000 3.0000 3.2000 3.2133 3.2142 3.2143	3.2143	6
$x = \frac{x}{100} + 3$	$\frac{1}{100}$	4.0000 3.0000 3.0300 3.0303	3.0303	4

$x = \frac{x}{200} + 3$	$\frac{1}{200}$	4.0000 3.0000 3.0150 3.0151	3.0151	4

5.2: Here we show the behaviour of the equation of Example 4.2. above.

$x^{f(e)} - x - 1 = 0$, $x = (x + 1)^{\sigma(e)}$, $x_0 = 0$, x_i^{**} =1.0035

	$\sigma(e)$	$\boldsymbol{Tx_m}$	Fixed point	m
$x^2 - x - 1 = 0$ $x = (x + 1)^{\frac{1}{2}}$	$\frac{1}{2}$	12.0000 1.0000 1.4142 1.5538 1.5981 1.6118 1.6161 1.6174 1.6179 1.6180	1.6180	10
$x^3 - x - 1 = 0$ $x = (x + 1)^{\frac{1}{3}}$	$\frac{1}{3}$	9.0000 1.0000 1.2599 1.3123 1.3224 1.3243 1.3246 1.3247	1.3247	8
$x^4 - x - 1 = 0$ $x = (x + 1)^{\frac{1}{4}}$	$\frac{1}{4}$	7.0000 1.0000 1.1892 1.2164 1.2201 1.2207	1.2207	6
$x^5 - x - 1 = 0$ $x = (x + 1)^{\frac{1}{5}}$	$\frac{1}{5}$	7.0000 1.0000 1.1487 1.1653 1.1671 1.1673	1.1673	6
$x^8 - x - 1 = 0$ $x = (x + 1)^{\frac{1}{8}}$	$\frac{1}{8}$	6.0000 1.0000 1.0905 1.0966 1.0970	1.0970	5
$x^{10} - x - 1 = 0$ $x = (x + 1)^{\frac{1}{10}}$	$\frac{1}{10}$	6.0000 1.0000 1.0718 1.0756 1.0758	1.0758	5
$x^{15} - x - 1 = 0$ $x = (x + 1)^{\frac{1}{15}}$	$\frac{1}{15}$	5.0000 1.0000 1.0473 1.0489 1.0490	1.0490	5

$x^{25} - x - 1 = 0$ $x = (x+1)^{\frac{1}{25}}$	$\frac{1}{25}$	5.0000 1.0000 1.0281 1.0287	1.0287	4
$x^{50} - x - 1 = 0$ $x = (x+1)^{\frac{1}{50}}$	$\frac{1}{50}$	4.0000 1.0000 1.0140 1.0141	1.0141	4
$x^{100} - x - 1 = 0$ $x = (x+1)^{\frac{1}{100}}$	$\frac{1}{100}$	4.0000 1.0000 1.0070	1.0070	3
$x^{200} - x - 1 = 0$ $x = (x+1)^{\frac{1}{200}}$	$\frac{1}{200}$	3.0000 1.0000 1.0035	1.0035	3

6.0. Conclusion

Using the idea of Banach, we proved that the fixed points for a family of functions of the type

$$\mathcal{F} := \{\{\sigma(e) * T(x)\}_{e\in E}\}_{x\in X}$$

where $\sigma: E \to [0,1)$ and $T: X \to X$ converges. We call such fixed point the limit fixed points (LFP).

With examples and computations, we showed the convergence.

7.0. Suggestion For Further Studies

Our research can be explored in the future by:

1. Investigating the effect of varied contraction constants of a sequence of self-maps on locally compact complete metric space on the sequence of fixed points.
2. Proposing an iterative scheme that approximates the Limit fixed point (LFP).

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