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Analysis of Single Variable Thick Plate Buckling Problems using Galerkin Method

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Abstract

A single variable shear deformable thick plate buckling equation is developed using systematic first principles approach. The equation is solved in closed form for simply supported boundary conditions using Galerkin method for in-plane uniaxial and biaxial compressive loads on the edges. The equation has one unknown and is similar in form as the thin plate equation, rendering it amenable to solution methods for the thin plate equation. The equation was derived using the total energy minimization method. The Galerkin method used exact sinusoidal shape functions of simply supported boundary conditions as the basis functions to construct the Galerkin variational integral which was minimized with respect to the unknown displacement parameters (amplitudes) to yield the governing equations of buckling. The eigenvalue problem was solved to find the zeros which gave the eigenvalues from which the critical buckling load was determined. Comparison of the critical buckling loads with the previous results in literature showed they were identical to the exact critical buckling results for simply supported plates for both cases of uniaxial and biaxial compressive loads considered for various ratios of dimension, a, to thickness h (a/h) and then dimension b to dimension a (b/a). The orthogonality of the eigenfunctions used as the solution basis simplified the resulting integration problem. The Galerkin methods gave exact results for the simply supported plate buckling problem because exact shape function which satisfied the Dirichlet boundary conditions were used, and the domain equations were also satisfied at all points on the plate. The novelty of this work is the first principle step by step approach used in the equilibrium method deployed in the derivation of the governing deferential equation of equilibrium (DEoE). Another unique feature of the study is the systematic formulation and solution of the Galerkin Variational Equation (GVE) for the plate problem considered.

1.0 Introduction

Three-dimensional (3D) structural members whose in-plane dimensions are much larger than their transverse dimensions, and which carry in-plane compressive loads in one or both of the directions are commonly encountered in engineering applications in the fields of civil, mechanical, aeronautical, marine and geotechnical engineering. Such plates are susceptible to buckling failures when certain critical in-plane loads are attained even when their material strengths had not been reached. It is therefore prudent that buckling analysis aimed at finding the least in-plane compressive force that can cause buckling failure is a vital component of their design. The behaviour of plates under in-plane compressive loads also depends on whether the plate is thin or thick. Thin plates are those plates with ratio of depth h, to width, a, (h/a) less than or equal to 0.05. When h/a is more than 0.05, but less than 0.10, the plate is moderately thick, and thick when h/a exceeds 0.10.

Thin plates have been extensively investigated and are governed by the Kirchhoff plate theory (KPT). KPT was derived using Kirchhoff hypothesis which essentially meant that the transverse shear strains which are responsible for distortions of the cross-sections are neglected. This limits the KPT and affects the accuracy of the theory when shear deformation effects become significant. The buckling analysis of thin plates has been extensively studied using a variety of analytical and numerical techniques. Onyia et al [1 – 3] studied elastic buckling analysis of thin rectangular plates using Galerkin-Kantorovich, single finite Fourier sine integral transform methods respectively. Onah et al [4] used single finite sine integral transform method for the stability solutions of CCSS thin plate under uniaxial compressive forces. Nwoji et al [5] used two-dimensional remnant of the finite sine integral transform method for stability solutions of simply supported thin plates.

Oguaghamba and Ike [6] used a Vlasov modification of the Galerkin method for the buckling analysis of thin SSSFr plates under uniaxial compressive loads, where the three edges are simply supported and one edge is free. Ike et al [7] utilized a generalized integral transformation method (GITM) for the buckling solutions of thin plates with two opposite clamped edges and two simply supported edges. Ike [8] used a combination of Ritz and Kantorovich methods for the accurate buckling solutions of clamped thin plates. In a recent study, Ike and Oguaghamba [9] used Kantorovich variational technique for the stability solutions of thin plates with two opposite simply supported edges.

Ike [10] used generalized integral transform method (GITM) to derive closed form bending solutions for fully clamped rectangular thin plates subjected to uniformly distributed transverse loads. In another study, Ike [11] used a Vlasov modification of the variational Kantorovich method to derive closed form solutions for flexural analysis problems of thin plates with simply supported and free edges (SFr SFr) subjected to distributed transverse loads. A previous study by Nwoji et al [12] implemented the Ritz variational method for the accurate flexural solutions of thin plates under hydrostatic load distributions. Ike [13] employed the Vlasov modification of the Galerkin variational method for developing accurate flexural solutions of rectangular thin plates.

The Kirchhoff hypothesis that assumes the orthogonality of lines normal to the middle surface of the plate before and during deformation implies that the Kirchhoff thin plate theory (KTP) as developed disregards transverse shear deformation. Hence, KPT is unable to correctly analyse the behaviours of thick plates, for which transverse shear deformations are significant. This led to the development of Reissner [14] plate theory (RPT) using a stress-based approach. Later, Mindlin [15] developed Mindlin plate theory (MPT) using a displacement formulation. Both RPT and MPT are first-order shear deformation theories yielding constant shear stress at the cross-section , and thus violating the shear stress-free conditions at the surfaces $(z = \pm 0.5h)$. Though shear correction factors k_s have been introduced to ensure the correct prediction of strain energy of shear deformation in line with elasticity theories, the lack of mathematically rigorous procedure for the determination of k_s renders the violation of shear-stress-free conditions at the surfaces a major issue of the RPT and MPT.

Nwoji et al [16] studied the bending analysis of simply supported rectangular Mindlin plates subjected to bisinusiodal loading using the method of double trigonometric series with undetermined parameters but did not consider buckling analysis. In more recent studies Ike [17] used the Ritz variational method to develop closed form analytical solutions for bending problems of clamped rectangular Mindlin plates subjected to uniformly distributed loading. The work did not however, consider buckling solutions. Ike [18] utilized the double Fourier series method (DFSM) to develop closed form bending solutions for simply supported rectangular Mindlin plates under uniformly distributed load and linearly distributed loading. Mindlin plate problems have been further studied by Ike [19,20,21], Ike et al [22] and Nwoji et al [23].

The search for improvements to the plate theories have led to further developments of plates using shear deformation theories of varying orders and refinements, primarily to ensure that the transverse shear stress-free conditions are satisfied at the edges and shear correction factors are not needed. Thus, trigonometric functions, hyperbolic functions, polynomial functions and exponential functions have been explored by several researchers as shear form functions leading to the construction of various shear deformation plate theories, higher order shear deformation plate theories and refined plate theories.

Ike [24] developed a third-order shear deformable plate bending model using first principles approach. The model was then solved by Navier's double trigonometric series method for the accurate flexural analysis of simply supported rectangular thick plate subjected to single sine loading, uniformly distributed load, linearly distributed load and enter point load. Onah et al [25] derived displacement and stress functions for three-dimensional elastostatic problems using systematic first principles approach. They applied their model to the accurate bending solutions of thick circular plates subjected to uniformly distributed loads.

Sayyad and Ghugal [26] used exponential shear deformation theory for the stability analysis of thick isotropic rectangular plates subjected to uniaxial and biaxial in-plane loads. Their formulation yielded a parabolic variation of the transverse shear strains γ_{xz}, γ_{yz} across the plate thickness. The transverse shear strains are zero at the top and bottom surfaces where $z = \pm 0.5h$. Virtual work principle was used to develop the governing differential equations of stability and the boundary conditions (BCs). They developed analytical solutions for simply supported (SSSS) square plates. Their study was validated with comparable buckling load results from previous studies.

Khalfi et al [27] studied the stability solutions of plates modeled using a sinusoidal shear deformation theory. Their work considered transverse shear strain effects, satisfied the transverse shear strain-free conditions at the plate surfaces and gave accurate buckling load solutions for both uniaxially loaded and biaxially loaded cases. They also derived accurate buckling solutions for simply supported istropic and orthotropic plates under uniaxial and biaxial loadings.

Mohseni and Naderi [28] innovatively used higher order shear and normal deformation plate theory (HOSNDT) to develop closed form solutions for the buckling of thick rectangular, porous plates made of functionally graded (FG) materials. The buckling equations were developed and solved to determine the critical buckling loads for simply supported BCs using Legendre's orthogonal polynomials and Navier's double trigonometric series method.

Ike [29] used double finite sine transformation method (DFSTM) to develop analytical closed-form solutions for uniaxial and biaxial buckling of single variable thick plate buckling problems. The formulation considered shear deformation and was a partial differential equation (PDE) expressed using an unknown transverse displacement variable. The DFSTM converted the PDE of the domain to an algebraic eigenvalue problem in the transformed space. The DFSTM was ideal for the simply supported thick plate solved because the sinusoidal kernel of the transformation satisfied the boundary conditions. The study gave exact solutions.

Gajbhiye et al [30] used a fifth order shear deformation plate theory for the stability investigations of thick rectangular plates under uniaxial and biaxial in-plane loads. They considered both transverse normal strain and transverse normal shear deformations. They assumed displacement field yielding non-linear variation of in-plane displacements and transverse shear strain variation with depth that satisfy the zero transverse shear strains at the surfaces. They obtained comparable buckling load parameters with previous results using first order shear deformation plate theory (FOSDPT), trigonometric shear deformation plate theory (TSDPT) and higher order shear deformation plate theory (HOSDPT).

Onyeka et al [31] used the total potential energy extremization method for the stability investigations of a thick rectangular isotropic plate under uniform uniaxial compressive loads. They assumed trigonometric displacement functions and considered transverse shear and in-plane shear deformation effects. Their obtained results agreed with previous results in the literature. Onyeka et al [32] used polynomial buckling displacements in the Ritz method for the buckling analysis of a rectangular thick plate clamped along two opposite edges, simply supported along the third edge and free along the fourth edge (SCFrC plate).

Onyeka et al [33] used direct variational method for the stability analysis of a clamped (CCCC) rectangular thick plate under uniaxial constant compressive load in the x direction. Polynomial displacement functions were used in the minimization of the total potential energy to find the critical buckling loads. Onyeka and Okeke [34] investigated the stability analysis of plate using one of the versions of the refined plate theory (RPT). In a related work, Onyeka et al [35] presented analytical solutions for the buckling of plates using direct variational techniques and minimizing the total energy of the buckling plate under uniaxial loading.

Onodagu et al [36] investigated the buckling analysis of simply supported thick rectangular plates assumed to be of linear elastic, homogeneous isotropic materials. The study assumed the buckling deflection as polynomial functions and used the Ritz method to develop the critical loads. In a related study, Godwin [37] used displacement buckling functions in the Ritz energy method for buckling load solutions of thick rectangular plates simply supported on three edges and free on one edge. The principle of minimization of the total potential energy of the thick plate buckling problem was used to develop solutions for the critical buckling load of SSFS plates. Their results were verified by favourable comparison with previous solutions in the literature. Timoshenko and Gere [38] presented closed form buckling solutions to the uniaxial and biaxial buckling of rectangular thin and thick plates.

In this work, a single variable thick plate buckling problem is formulated using first principles equilibrium approach. The Galerkin method is then used to obtain exact analytical solutions to the partial differential equation formulated for the case of simply supported edges and when the plate is subjected to uniaxial uniform compressive forces in one direction; and biaxial uniform compressive forces in both x and y directions.

2. Theoretical Framework

The assumptions are as follows:

(i) the displacements are infinitesimally small relative to the plate depth, h, and the resulting strains are infinitesimal.

(ii) the transverse displacement in the middle surface ($z = 0$) is the sum of transverse displacement components due to flexure w_f and shear deformation w_s. Both transverse displacement components are functions of the in-plane coordinates x, y only and don't vary with z. Thus,

$$w(x,y,z) = w(x,y,z=0) = w_f(x,y,z=0) + w_s(x,y,z=0) = w_f(x,y) + w_s(x,y)$$

(iii) the transverse normal stress σ_{zz} is so small compared with the in-plane normal stresses σ_{xx}, σ_{yy} and can be neglected. $\sigma_{zz} = 0$

Displacements
The displacement field components in the *x*, *y* and *z* directions are:

$$u(x,y,z) = u(x,y,z=0) = -z\frac{\partial w_f}{\partial x}, \quad v(x,y,z) = v(x,y,z=0) = -z\frac{\partial w_f}{\partial y},$$
$$w(x,y,z) = w(x,y,z=0) = w(x,y) = w_f(x,y) + w_s(x,y) \qquad (1)$$

where: w_f is the flexural component of $w(x, y)$, $w_s(x, y)$ is the shear component of $w(x, y)$.

Strains
The normal strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ are found using the kinematic equations of small displacement linear elasticity theory as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w_f}{\partial x^2}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w_f}{\partial y^2} \qquad (2)$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

The shear strains γ_{xy} and transverse shear strains γ_{xz}, γ_{yz} are similarly found using Equation (1) in the kinematic equations of small displacement linear elastic theory as:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w_f}{\partial x \partial y}$$
$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial w_s}{\partial x} \qquad (3)$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial w_s}{\partial y}$$

Stresses
The normal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ and shear stresses $\tau_{xy}, \tau_{xz}, \tau_{yz}$ are obtained using the stress-strain laws for plane stress elasticity of homogeneous, isotropic materials.

$$\sigma_{xx} = \frac{E}{1-\mu^2}(\varepsilon_{xx} + \mu\varepsilon_{yy})$$
$$\sigma_{yy} = \frac{E}{1-\mu^2}(\varepsilon_{yy} + \mu\varepsilon_{xx})$$
$$\sigma_{zz} = 0 \qquad (4)$$
$$\tau_{xy} = G\gamma_{xy}$$
$$\tau_{xz} = G\gamma_{xz}$$
$$\tau_{yz} = G\gamma_{yz}$$

wherein *E* is the Young's modulus of elasticity, *G* is the shear modulus, μ is the Poisson's ratio

$$G = \frac{E}{2(1+\mu)} \quad (5)$$

Hence, substitution of the strains Equations (2) and (3) into Equation (4) and simplifying,

$$\sigma_{xx} = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w_f}{\partial x^2} + \mu\frac{\partial^2 w_f}{\partial y^2}\right)$$
$$\sigma_{yy} = -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w_f}{\partial y^2} + \mu\frac{\partial^2 w_f}{\partial x^2}\right) \quad (6)$$
$$\tau_{xy} = -2zG\frac{\partial^2 w_f}{\partial x\partial y} = \frac{-E}{1-\mu^2}(1-\mu)\frac{\partial^2 w_f}{\partial x\partial y}$$
$$\tau_{xz} = G\frac{\partial w_s}{\partial x} = \frac{E}{1-\mu^2}\left(\frac{1-\mu}{2}\right)\frac{\partial w_s}{\partial x}$$
$$\tau_{yz} = \frac{\partial w_s}{\partial y} = \frac{E}{1-\mu^2}\left(\frac{1-\mu}{2}\right)\frac{\partial w_s}{\partial y}$$

Introducing transverse shear stress correction factor, k_s to correct the violation of transverse shear stress-free conditions at the plate surfaces $z = \pm 0.5h$, gives:

$$\tau_{xz} = \frac{E}{1-\mu^2}\left(\frac{1-\mu}{2}\right)k_s\frac{\partial w_s}{\partial x} = Gk_s\frac{\partial w_s}{\partial x} \quad (7a)$$
$$\tau_{yz} = \frac{E}{1-\mu^2}\left(\frac{1-\mu}{2}\right)k_s\frac{\partial w_s}{\partial y} = Gk_s\frac{\partial w_s}{\partial y} \quad (7b)$$

Internal Stress Resultants (Bending moments, M_{xx}, M_{yy}; Twisting moment, M_{xy} and Shear forces Q_x, Q_y)
The bending moments M_{xx}, M_{yy} are:

$$M_{xx} = \int_{-h/2}^{h/2}\sigma_{zz}z\,dz = -\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz\left(\frac{\partial^2 w_f}{\partial x^2} + \mu\frac{\partial^2 w_f}{\partial y^2}\right) = -D\left(\frac{\partial^2 w_f}{\partial x^2} + \mu\frac{\partial^2 w_f}{\partial y^2}\right) \quad (8)$$
$$M_{yy} = \int_{-h/2}^{h/2}\sigma_{zz}z\,dz = -\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz\left(\frac{\partial^2 w_f}{\partial y^2} + \mu\frac{\partial^2 w_f}{\partial x^2}\right) = -D\left(\frac{\partial^2 w_f}{\partial y^2} + \mu\frac{\partial^2 w_f}{\partial x^2}\right) \quad (9)$$

where $D = \frac{Eh^3}{12(1-\mu^2)}$ (10)

D is the modulus of flexural rigidity of the plate.
The twisting moment M_{xy} is:

$$M_{xy} = \int_{-h/2}^{h/2}\tau_{xy}z\,dz = -\int_{-h/2}^{h/2}\frac{Ez^2}{1-\mu^2}dz(1-\mu)\frac{\partial^2 w_f}{\partial x\partial y} = -D(1-\mu)\frac{\partial^2 w_f}{\partial x\partial y} \quad (11)$$

The shear force distributions Q_x, Q_y are:

$$Q_x = \int_{-h/2}^{h/2}\tau_{xz}dz = \int_{-h/2}^{h/2}\frac{Ek_s}{2(1+\mu)}\frac{\partial w_s}{\partial x}dz = \frac{Ehk_s}{2(1+\mu)}\frac{\partial w_s}{\partial x} \quad (12)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz = \int_{-h/2}^{h/2} \frac{Ek_s}{2(1+\mu)} \frac{\partial w_s}{\partial y} dz = \frac{Ehk_s}{2(1+\mu)} \frac{\partial w_s}{\partial y} \quad (13)$$

Differential Equations of Equilibrium (DEoE)
The gross DEoE are:

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (14a)$$

$$\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \quad (14b)$$

$$-\frac{\partial Q_x}{\partial x} - \frac{\partial Q_y}{\partial y} - q + N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} = 0 \quad (14c)$$

where N_{xx}, N_{yy}, N_{xy} are the in-plane forces.
Hence from Equation (14)

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -D\frac{\partial}{\partial x}\nabla^2 w_f \quad (15a)$$

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -D\frac{\partial}{\partial y}\nabla^2 w_f \quad (15b)$$

Also, from Equations (7a) and (7b)

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz = \int_{-h/2}^{h/2} Gk_s \frac{\partial w_s}{\partial x} dz = Ghk_s \frac{\partial w_s}{\partial x} \quad (16a)$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz = \int_{-h/2}^{h/2} Gk_s \frac{\partial w_s}{\partial y} dz = Ghk_s \frac{\partial w_s}{\partial y} \quad (16b)$$

Hence from Equations (15a) and (16a)

$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w_f = Ghk_s \frac{\partial w_s}{\partial x} \quad (17a)$$

Similarly, from Equations (15b) and (16b)

$$Q_y = -D\frac{\partial}{\partial y}\nabla^2 w_f = Ghk_s \frac{\partial w_s}{\partial y} \quad (17b)$$

Integrating Equations (17a) and (17b) gives:

$$w_s = \frac{-D}{Ghk_s}\nabla^2 w_f = \frac{-h^2}{6k_s(1-\mu)}\nabla^2 w_f \quad (18)$$

Hence,

$$w(x, y) = w_f - \frac{D}{Ghk_s}\nabla^2 w_f \quad (19)$$

From Equation (14c),

$$-\frac{\partial}{\partial x}\left(\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial M_{yx}}{\partial x}+\frac{\partial M_{yy}}{\partial y}\right)-q+N_{xx}\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \qquad (20)$$

$$-\frac{\partial^2 M_{xx}}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}-\frac{\partial^2 M_{yy}}{\partial y^2}-q+N_{xx}\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \qquad (21)$$

$$-\frac{\partial^2}{\partial x^2}\left(-D\left(\frac{\partial^2 w_f}{\partial x^2}+\mu\frac{\partial^2 w_f}{\partial y^2}\right)\right)-2\frac{\partial^2}{\partial x\partial y}\left(-D(1-\mu)\frac{\partial^2 w_f}{\partial x\partial y}\right)-$$

$$\frac{\partial^2}{\partial y^2}\left(-D\left(\frac{\partial^2 w_f}{\partial y^2}+\mu\frac{\partial^2 w_f}{\partial x^2}\right)\right)-q+N_{xx}\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \qquad (22)$$

$$D\left(\frac{\partial^4 w_f}{\partial x^4}+2\frac{\partial^4 w_f}{\partial x^2\partial y^2}+\frac{\partial^4 w_f}{\partial y^4}\right)-q+N_{xx}\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \qquad (23)$$

Substituting the equation for w in terms of w_f in Equation (23) gives the single variable thick plate equation as:

$$D\nabla^4 w_f+N_{xx}\frac{\partial^2}{\partial x^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)+2N_{xy}\frac{\partial^2}{\partial x\partial y}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)+$$

$$N_{yy}\frac{\partial^2}{\partial y^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)=q \qquad (24)$$

For buckling in the absence of transverse loads, $q(x, y) = 0$ and Equation (24) simplifies to:

$$D\nabla^4 w_f+N_{xx}\frac{\partial^2}{\partial x^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)+2N_{xy}\frac{\partial^2}{\partial x\partial y}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)+$$

$$N_{yy}\frac{\partial^2}{\partial y^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)=0 \qquad (25)$$

When $N_{xy} = 0$,

$$\nabla^4 w_f+\frac{N_{xx}}{D}\frac{\partial^2}{\partial x^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)+\frac{N_{yy}}{D}\frac{\partial^2}{\partial y^2}\left(w_f-\frac{D}{Ghk_s}\nabla^2 w_f\right)=0 \qquad (26)$$

3. Methodology

This study considers the buckling of thick plates simply supported along all the edges $x = 0$, $x = a$, $y = 0$, $y = b$ and subjected to:

(i) Case 1: uniaxial uniform compressive load N_{xx} in the x direction shown in Figure 1.
(ii) Case 2: uniaxial uniform compressive load N_{yy} in the y direction shown in Figure 2, and
(iii) Case 3: biaxial compressive load $N_{xx} = N_{yy} = N_0$ in both x and y directions as shown in Figure 3.

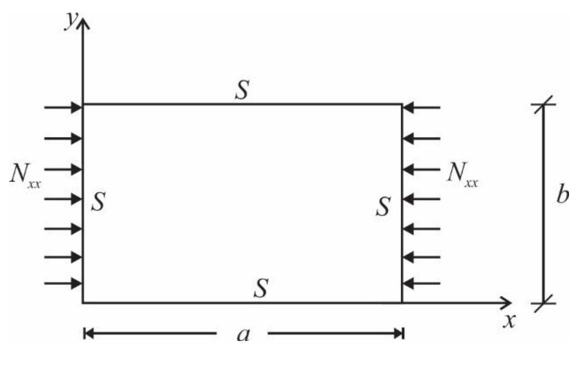

Figure 1: Simply supported thick plate under uniaxial uniform compressive load, N_{xx} (in the x direction)

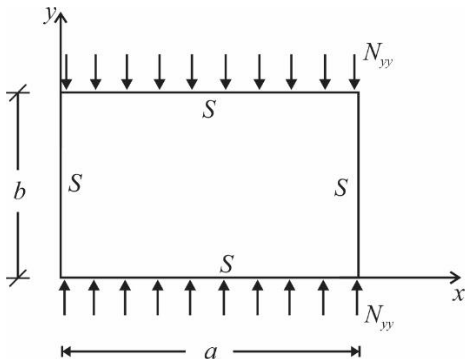

Figure 2: Simply supported thick plate subjected to uniaxial uniform compressive load, N_{yy} (in the y direction)

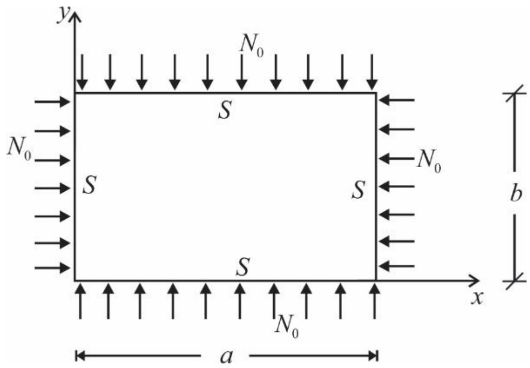

Figure 3: Simply supported thick plate subjected to biaxial uniform compressive load, N_0

The boundary conditions are:

$$w(0, y) = \left(w_f - \frac{D}{Ghk_s} \nabla^2 w_f \right)\Bigg|_{x=0, y} = 0 \tag{27}$$

$$M_x(0, y) = -D\left(\frac{\partial^2 w_f}{\partial x^2} + \mu \frac{\partial^2 w_f}{\partial y^2} \right)\Bigg|_{x=0, y} = 0 \tag{28}$$

Hence, $w_f(x = 0, y) = 0$

$$\frac{\partial^2 w_f}{\partial x^2}\Bigg|_{x=0, y} = 0 \tag{29}$$

Similarly, $w_f(x = a, y) = 0$

$$\frac{\partial^2 w_f}{\partial x^2}(x = a, y) = 0 \tag{30}$$

$$w_f(x, y = 0) = 0$$

$$\left.\frac{\partial^2 w_f}{\partial y^2}\right|_{(x,y=0)} = 0 \quad (31)$$

$$w_f(x, y = b) = 0$$

$$\frac{\partial^2 w_f}{\partial y^2}(x, y = b) = 0 \quad (32)$$

The deflection $w_f(x, y)$ is expressed using suitable exact buckling shape function that satisfies the boundary conditions as:

$$w_f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{f_{mn}} \sin\alpha_m x \sin\beta_n y \quad (33)$$

where $\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}$ (34)

The Galerkin Variational Equation (GVE) for the three cases are; for case 1:

$$\int_0^b\int_0^a\left\{\nabla^4\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y + \frac{N_{xx}}{D}\left[\frac{\partial^2}{\partial x^2}\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x \sin\beta_n y - \frac{D}{Ghk_s}\frac{\partial^2}{\partial x^2}\nabla^2\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y\right]\right\}\sin\bar{\alpha}_m x \sin\bar{\beta}_n y\, dxdy = 0 \quad (35)$$

where $\bar{\alpha}_m = \frac{\bar{m}\pi}{a}, \quad \bar{\beta}_n = \frac{\bar{n}\pi}{b}$ (36)

For case 2, the GVE is:

$$\int_0^b\int_0^a\left\{\nabla^4\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y + \frac{N_{yy}}{D}\left[\frac{\partial^2}{\partial y^2}\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x \sin\beta_n y - \frac{D}{Ghk_s}\frac{\partial^2}{\partial y^2}\nabla^2\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y\right]\right\}\sin\bar{\alpha}_m x \sin\bar{\beta}_n y\, dxdy = 0 \quad (37)$$

For case 3, biaxial compressive loading N_0, the GVE is:

$$\int_0^b\int_0^a\left\{\nabla^4\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y + \frac{N_0}{D}\left[\nabla^2\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x \sin\beta_n y - \frac{D}{Ghk_s}\left(\frac{\partial^2}{\partial x^2}\nabla^2\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y\right) + \frac{\partial^2}{\partial y^2}\nabla^2\sum_m^{\infty}\sum_n^{\infty} w_{f_{mn}} \sin\alpha_m x\sin\beta_n y\right]\right\}\sin\bar{\alpha}_m x \sin\bar{\beta}_n y\, dxdy = 0 \quad (38)$$

Simplifying Equation (35) gives:

$$\sum_{m}^{\infty}\sum_{n}^{\infty}(\alpha_m^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4)w_{f_{mn}}I_{mn} + \frac{N_{xx}}{D}\left(-\alpha_m^2 w_{f_{mn}}I_{mn} - \frac{D}{Ghk_s}(\alpha_m^4 + \beta_n^2\alpha_m^2)w_{f_{mn}}I_{mn}\right) = 0 \quad (39)$$

where

$$I_{mn} = \int_0^b\int_0^a(\sin\alpha_m x\sin\beta_n y\sin\bar{\alpha}_m x\sin\bar{\beta}_n y)dxdy \quad (40)$$

$$I_{mn} = \int_0^b\int_0^a\sin^2\alpha_m x\sin^2\beta_n y\,dxdy \quad (41)$$

$$I_{mn} = \frac{ab}{4} \qquad \text{if } m = \bar{m},\ n = \bar{n} \quad (42)$$

$$I_{mn} = 0 \qquad \text{if } m \neq \bar{m},\ n \neq \bar{n} \quad (43)$$

Case 2, the GVE is Equation (37), which is simplified as:

$$\sum_{m}^{\infty}\sum_{n}^{\infty}\int_0^b\int_0^a\left\{\nabla^4 w_{f_{mn}}\sin\alpha_m x\sin\beta_n y + \frac{N_{yy}}{D}\left[\frac{\partial^2}{\partial y^2}w_{f_{mn}}\sin\alpha_m x\sin\beta_n y - \frac{D}{Ghk_s}\left(\frac{\partial^2}{\partial y^2}\nabla^2 w_{f_{mn}}\sin\alpha_m x\sin\beta_n y\right)\right]\right\}\sin\bar{\alpha}_m x\sin\bar{\beta}_n y\,dxdy = 0 \quad (44)$$

Hence,

$$\sum_{m}^{\infty}\sum_{n}^{\infty}w_{f_{mn}}\left\{(\alpha_m^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4)I_{mn} + \frac{N_{yy}}{D}\left(-\beta_n^2 I_{mn} - \frac{D}{Ghk_s}(\alpha_m^2\beta_n^2 + \beta_n^4)I_{mn}\right)\right\} = 0 \quad (45)$$

Hence, for nontrivial solutions,

$$(\alpha_m^2 + \beta_n^2)^2 - \frac{N_{yy}}{D}\left(\beta_n^2 + \frac{D}{Ghk_s}(\beta_n^4 + \alpha_m^2\beta_n^2)\right) = 0 \quad (46)$$

For Case 3, the GVE is from simplifying Equation (38),

$$\sum_{m}^{\infty}\sum_{n}^{\infty}w_{f_{mn}}\left\{\left(\alpha_m^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4\right)I_{mn} + \frac{N_0}{D}\left(-(\alpha_m^2 + \beta_n^2)I_{mn} - \frac{D}{Ghk_s}(\alpha_m^4 + \alpha_m^2\beta_n^2)I_{mn} + (\alpha_m^2\beta_n^2 + \beta_n^4)I_{mn}\right)\right\} = 0 \quad (47)$$

Hence, for nontrivial solutions,

$$(\alpha_m^2 + \beta_n^2)^2 - \frac{N_0}{D}\left((\alpha_m^2 + \beta_n^2) + \frac{D}{Ghk_s}(\alpha_m^4 + 2\alpha_m^2\beta_n^2 + \beta_n^4)\right) = 0 \quad (48)$$

4. Results and Discussion

For case 1, Equation (39) becomes:

$$(\alpha_m^2 + \beta_n^2)^2 w_{f_{mn}} + \frac{N_{xx}}{D}\left(-\alpha_m^2 w_{f_{mn}} - \frac{D}{Ghk_s}\left(\alpha_m^2(\alpha_m^2 + \beta_n^2)\right)w_{f_{mn}}\right) \quad (49)$$

$$(\alpha_m^2 + \beta_n^2)^2 + \frac{N_{xx}}{D}\left(-\alpha_m^2 - \frac{D}{Ghk_s}\left(\alpha_m^4 + \alpha_m^2\beta_n^2\right)\right) = 0 \quad (50)$$

For nontrivial solutions, $w_{f_{mn}} \neq 0$.

Hence,

$$\frac{N_{xx}}{D}\left(\alpha_m^2 + \frac{D}{Ghk_s}\left(\alpha_m^4 + \alpha_m^2\beta_n^2\right)\right) = (\alpha_m^2 + \beta_n^2)^2 \quad (51)$$

$$\frac{N_{xx}}{D} = \frac{(\alpha_m^2 + \beta_n^2)^2}{\alpha_m^2 + \frac{D}{Ghk_s}\left(\alpha_m^4 + \alpha_m^2\beta_n^2\right)} \quad (52)$$

$$\frac{N_{xx}}{D} = \frac{\left(\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right)}{\left(\frac{m\pi}{a}\right)^2 + \frac{h^2}{6(1-\mu)k_s}\left(\left(\frac{m\pi}{a}\right)^4 + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2\right)} \quad (53)$$

$$\frac{N_{xx}}{D} = \frac{\pi^4\left(\frac{m^4}{a^4} + 2\frac{m^2n^2}{a^2b^2} + \frac{n^4}{b^4}\right)}{\frac{m^2\pi^2}{a^2} + \frac{h^2\pi^4}{6(1-\mu)k_s}\left(\frac{m^4}{a^4} + \frac{m^2n^2}{a^2b^2}\right)} \quad (54)$$

$$\frac{N_{xx}}{D} = \frac{\pi^4\left(m^4 + 2m^2n^2\frac{a^2}{b^2} + n^4\frac{a^4}{b^4}\right)}{a^2m^2\pi^2 + \frac{\pi^4}{6(1-\mu)k_s}h^2\left(m^4 + m^2n^2\frac{a^2}{b^2}\right)} \quad (55)$$

$$\frac{N_{xx}}{D} = \frac{\pi^4\left(m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4\right)}{\frac{a^2m^2\pi^4}{\pi^2} + \frac{\pi^4}{6(1-\mu)k_s}h^2\left(m^4 + m^2n^2\left(\frac{a}{b}\right)^2\right)} \quad (56)$$

$$\frac{N_{xx}}{D} = \frac{m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4}{\frac{a^2m^2}{\pi^2} + \frac{h^2}{6(1-\mu)k_s}\left(m^4 + m^2n^2\left(\frac{a}{b}\right)^2\right)} \quad (57)$$

$$\frac{N_{xx}}{D} = \frac{m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4}{a^2\left(\frac{m^2}{\pi^2} + \frac{1}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^4 + m^2n^2\left(\frac{a}{b}\right)^2\right)\right)} \quad (58)$$

$$\frac{N_{xx}}{D} = \frac{m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4}{\frac{a^2}{\pi^2}\left(m^2 + \frac{1}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^4\pi^2 + m^2n^2\pi^2\left(\frac{a}{b}\right)^2\right)\right)} \quad (59)$$

$$\frac{N_{xx}a^2}{\pi^2 D} = \frac{m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4}{m^2 + \frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^4 + m^2n^2\left(\frac{a}{b}\right)^2\right)} \quad (60)$$

$$N_{xx_{cr}} = N_{xx}(m = 1, n = 1) \quad (61)$$

$$\frac{N_{xx_{cr}}a^2}{\pi^2 D} = \frac{1 + 2\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4}{1 + \frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(1 + \left(\frac{a}{b}\right)^2\right)} \quad (62)$$

For $k_s = 5/6$,

$$N_{xx_{cr}} = \frac{D\pi^2}{a^2}\left(\frac{1 + 2\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4}{1 + \frac{\pi^2}{5(1-\mu)}\left(\frac{h}{a}\right)^2\left(1 + \left(\frac{a}{b}\right)^2\right)}\right) = K_{cr}\left(\frac{a}{b}\right)\frac{D\pi^2}{a^2} \quad (63)$$

$$K_{cr}\left(\frac{a}{b}, \frac{h}{a}\right) = \frac{a^2 N_{xx_{cr}}}{\pi^2 D} = \frac{1 + 2\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4}{1 + \frac{\pi^2}{5(1-\mu)}\left(\frac{h}{a}\right)^2\left(1 + \left(\frac{a}{b}\right)^2\right)} \quad (64)$$

Solving for case 2, gives:

$$\frac{N_{yy}}{D} = \frac{(\alpha_m^2 + \beta_n^2)^2}{\beta_n^2 + \frac{D}{Ghk_s}(\alpha_m^2\beta_n^2 + \beta_n^4)} \quad (65)$$

$$\frac{N_{yy}}{D} = \frac{\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4}{\left(\frac{n\pi}{b}\right)^2 + \frac{D}{Ghk_s}\left(\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right)} \quad (66)$$

$$\frac{N_{yy}}{D} = \frac{\pi^4\left(\frac{m^4}{a^4} + 2\frac{m^2n^2}{a^2b^2} + \frac{n^4}{b^4}\right)}{\pi^2\left(\frac{n^2}{b^2}\right) + \frac{D\pi^4}{Ghk_s}\left(\frac{m^2n^2}{a^2b^2} + \frac{n^4}{b^4}\right)} \quad (67)$$

$$\frac{N_{yy}}{D} = \frac{\pi^4\left(m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4\right)}{\pi^2\left\{\left(\frac{n^2a^4}{b^2\pi^2}\right) + \frac{h^2}{6(1-\mu)k_s}\left(m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4\right)\right\}} \quad (68)$$

$$\frac{N_{yy}}{D} = \frac{m^4 + 2m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4}{a^2\left\{\frac{n^2}{\pi^2}\left(\frac{a}{b}\right)^2 + \frac{1}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^2n^2\left(\frac{a}{b}\right)^2 + n^4\left(\frac{a}{b}\right)^4\right)\right\}} \quad (69)$$

$$\frac{N_{yy}}{D}=\frac{m^4+2m^2n^2\left(\frac{a}{b}\right)^2+n^4\left(\frac{a}{b}\right)^4}{\frac{a^2}{\pi^2}\left\{n^2\left(\frac{a}{b}\right)^2+\frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^2n^2\left(\frac{a}{b}\right)^2+n^4\left(\frac{a}{b}\right)^4\right)\right\}} \quad (70)$$

$$\frac{N_{yy}a^2}{D\pi^2}=\frac{m^4+2m^2n^2\left(\frac{a}{b}\right)^2+n^4\left(\frac{a}{b}\right)^4}{n^2\left(\frac{a}{b}\right)^2+\frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^2n^2\left(\frac{a}{b}\right)^2+n^4\left(\frac{a}{b}\right)^4\right)} \quad (71)$$

$$\frac{N_{yy_{cr}}a^2}{D\pi^2}=\frac{1+2\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^4}{\left(\frac{a}{b}\right)^2+\frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^4\right)} \quad (72)$$

$$\frac{N_{yy_{cr}}a^2}{D\pi^2}=K_{yy_{cr}}\left(\frac{a}{b},\frac{h}{a}\right) \quad (73)$$

For case 3, biaxial compression load, N_0, the solution is:

$$\frac{N_0}{D}=\frac{(\alpha_m^2+\beta_n^2)^2}{\alpha_m^2+\beta_n^2+\frac{D}{Ghk_s}(\alpha_m^2+\beta_n^2)^2} \quad (74)$$

Dividing by $\alpha_m^2+\beta_n^2$ gives:

$$\frac{N_0}{D}=\frac{\alpha_m^2+\beta_n^2}{1+\frac{D}{Ghk_s}(\alpha_m^2+\beta_n^2)} \quad (75)$$

$$\frac{N_0}{D}=\frac{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}{1+\frac{h^2}{6(1-\mu)k_s}\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)} \quad (76)$$

$$\frac{N_0}{D}=\frac{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}{1+\frac{h^2\pi^2}{6(1-\mu)k_s}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \quad (77)$$

$$\frac{N_0}{D}=\frac{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}{\pi^2\left(\frac{1}{\pi^2}+\frac{h^2}{6(1-\mu)k_s}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right)} \quad (78)$$

$$\frac{N_0}{D}=\frac{\left(m^2+n^2\left(\frac{a}{b}\right)^2\right)}{\frac{a^2}{\pi^2}+\frac{h^2}{6(1-\mu)k_s}\left(m^2+n^2\left(\frac{a}{b}\right)^2\right)} \quad (79)$$

$$\frac{N_0}{D} = \frac{m^2 + n^2\left(\frac{a}{b}\right)^2}{\frac{a^2}{\pi^2}\left(1 + \frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^2 + n^2\left(\frac{a}{b}\right)^2\right)\right)} \quad (80)$$

$$\frac{N_0 a^2}{D\pi^2} = \frac{m^2 + n^2\left(\frac{a}{b}\right)^2}{1 + \frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(m^2 + n^2\left(\frac{a}{b}\right)^2\right)} \quad (81)$$

For $m = 1$, $n = 1$,

$$\frac{N_{0_{cr}} a^2}{D\pi^2} = \frac{1 + \left(\frac{a}{b}\right)^2}{1 + \frac{\pi^2}{6(1-\mu)k_s}\left(\frac{h}{a}\right)^2\left(1 + \left(\frac{a}{b}\right)^2\right)} \quad (82)$$

For $k_s = 5/6$,

$$\frac{N_{0_{cr}} a^2}{D\pi^2} = \frac{1 + \left(\frac{a}{b}\right)^2}{1 + \frac{\pi^2}{5(1-\mu)}\left(\frac{h}{a}\right)^2\left(1 + \left(\frac{a}{b}\right)^2\right)} \quad (83)$$

Table 1
Dimensionless critical buckling load coefficients of square plate under uniaxial compressive load for $\mu = 0.30$, $E =$ 210GPa, $K_{cr} = \frac{a^2 N_{xx_{cr}}}{\pi^2 D}$, $\frac{N_{xx_{cr}} a^2}{Eh^3} = \bar{N}$

a/h	Reference		$\bar{N}$
5	Present	$k_s = 2/3$	2.8200
	Present	$k_s = 5/6$	2.9498
	Present	$k_s = 1$	3.0432
	Khalfi et al [27]	$k_s = 2/3$	2.8200
	Khalfi et al [27]	$k_s = 5/6$	2.9498
	Khalfi et al [27]	$k_s = 1$	3.0432
10	Present	$k_s = 2/3$	3.3772
	Present	$k_s = 5/6$	3.4222
	Present	$k_s = 1$	3.4530
	Khalfi et al [27]	$k_s = 2/3$	3.3772
	Khalfi et al [27]	$k_s = 5/6$	3.4222
	Khalfi et al [27]	$k_s = 1$	3.4530
20	Present	$k_s = 2/3$	3.5556
	Present	$k_s = 5/6$	3.5650
	Present	$k_s = 1$	3.5733
	Khalfi et al [27]	$k_s = 2/3$	3.5556
	Khalfi et al [27]	$k_s = 5/6$	3.5650
	Khalfi et al [27]	$k_s = 1$	3.5733
50	Present	$k_s = 2/3$	3.6051
	Present	$k_s = 5/6$	3.6071

	Present	$k_s = 1$	3.6085
	Khalfi et al [27]	$k_s = 2/3$	3.6051
	Khalfi et al [27]	$k_s = 5/6$	3.6071
	Khalfi et al [27]	$k_s = 1$	3.6085
100	Present	$k_s = 2/3$	3.6127
	Present	$k_s = 5/6$	3.6132
	Present	$k_s = 1$	3.6135
	Khalfi et al [27]	$k_s = 2/3$	3.6127
	Khalfi et al [27]	$k_s = 5/6$	3.6132
	Khalfi et al [27]	$k_s = 1$	3.6135
	KPT (Timoshenko and Gere [38])		3.6152

Table 1 is shown plotted as $\bar{N}$ vs a/h (for $k_s = 5/6$) in Figure 4. Figure 4 shows that the present results coincide with previous results by Khalfi et al [27].

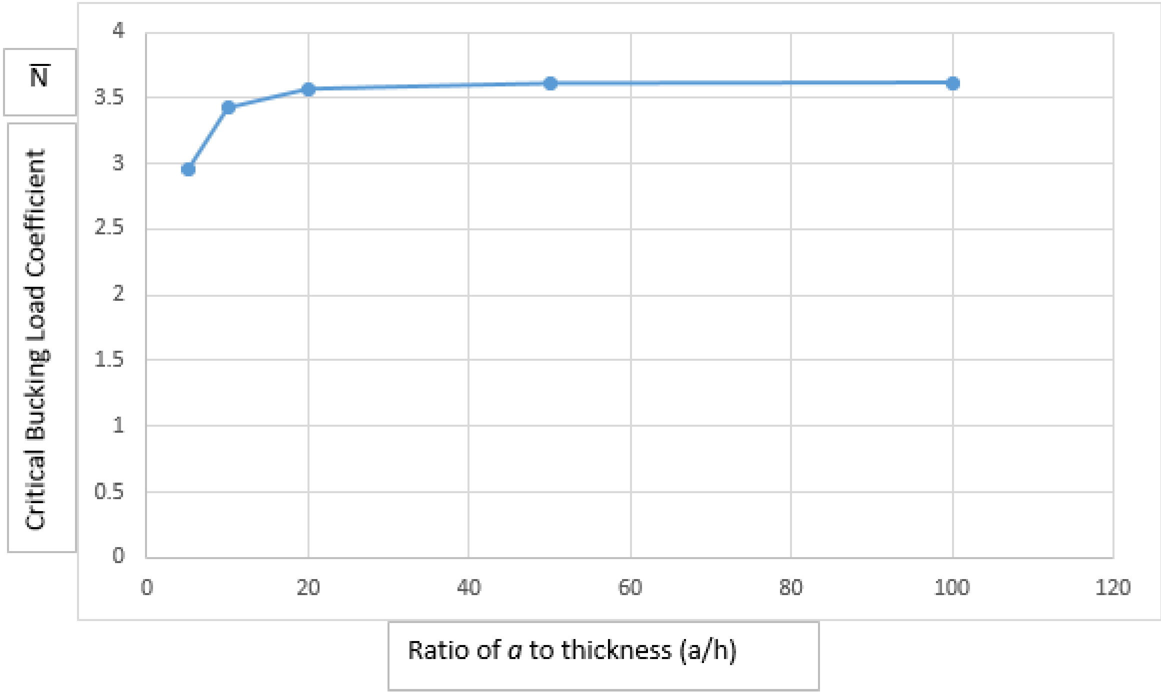

Figure 4: Plot of $\bar{N}$ vs $^a/_h$ for uniaxial buckling of plate for $k_s = {}^5/_6$

Table 2
Comparison of dimensionless critical buckling load factors for biaxial compression load, for $\mu = 0.30$, $E = 210$GPa, $\bar{N} = \frac{N_{cr}a^2}{Eh^3}$

$^a/_h$	Reference		$\bar{N}$
5	Present	$k_s = 2/3$	1.410
	Present	$k_s = 5/6$	1.4749
	Present	$k_s = 1$	1.5218
	Khalfi et al [27]	$k_s = 2/3$	1.410
	Khalfi et al [27]	$k_s = 5/6$	1.4749
	Khalfi et al [27]	$k_s = 1$	1.5218
10	Present	$k_s = 2/3$	1.6886
	Present	$k_s = 5/6$	1.7111
	Present	$k_s = 1$	1.7265

	Khalfi et al [27]	$k_s = 2/3$	1.6886
	Khalfi et al [27]	$k_s = 5/6$	1.7111
	Khalfi et al [27]	$k_s = 1$	1.7265
20	Present	$k_s = 2/3$	1.7763
	Present	$k_s = 5/6$	1.7825
	Present	$k_s = 1$	1.7866
	Khalfi et al [27]	$k_s = 2/3$	1.7763
	Khalfi et al [27]	$k_s = 5/6$	1.7825
	Khalfi et al [27]	$k_s = 1$	1.7866
50	Present	$k_s = 2/3$	1.8025
	Present	$k_s = 5/6$	1.8036
	Present	$k_s = 1$	1.8042
	Khalfi et al [27]	$k_s = 2/3$	1.8025
	Khalfi et al [27]	$k_s = 5/6$	1.8036
	Khalfi et al [27]	$k_s = 1$	1.8042
100	Present	$k_s = 2/3$	1.8063
	Present	$k_s = 5/6$	1.8066
	Present	$k_s = 1$	1.8068
	Khalfi et al [27]	$k_s = 2/3$	1.8063
	Khalfi et al [27]	$k_s = 5/6$	1.8066
	Khalfi et al [27]	$k_s = 1$	1.8068
	KPT (Timoshenko and Gere [38])		1.8076

Table 2 is plotted as $\bar{N}_0$ vs a/h (for k_s = 2/3) in Figure 5. Figure 5 shows that the present results are identical with previous results by Khalfi et al [27].

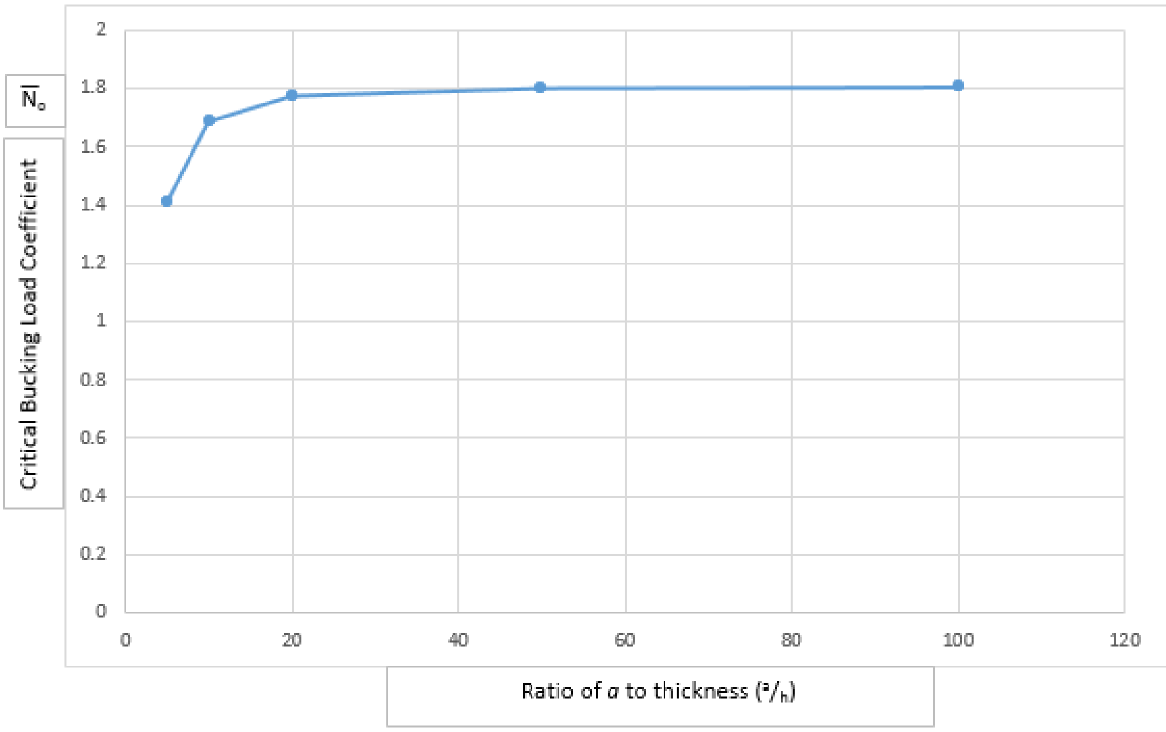

Figure 5: Plot of $\bar{N}_o$ vs $^a/_h$ for biaxial buckling of plate for $k_s = {}^2/_3$

5. Conclusion

A single variable shear deformable plate buckling formulation has been derived in this paper using first principles approach. The equation has one variable and is similar to the equation of the thin plate theory. Galerkin method was used to derive analytical solutions for the case of simply supported plates under uniaxial and biaxial compressive loads. The conclusions are:

(i) the single variable shear deformable plate equation developed accounts for transverse shear deformation effects, and yields transverse shear stress-free conditions at the surfaces $z = \pm 0.5h$
(ii) the formulation accurately gives the critical buckling loads of isotropic rectangular plates for various a/h and b/a ratios.
(iii) critical buckling load increased as the aspect ratio increased.
(iv) this work is limited to thick plates that are isotropic and linearly plastic, and hence does not consider material nonlinearity.

Notations/Nomenclature

x, y	in-plane coordinates
z	transverse coordinate
$a,\ b$	in-plane dimensions
u	in-plane displacement component in the x direction
v	in-plane displacement component in the y direction
w	transverse displacement component in the z direction
w_f	flexural component of transverse displacement
w_s	shear component of transverse displacement
ε_{xx}	in-plane normal strain in the x direction
ε_{yy}	in-plane normal strain in the y direction
ε_{zz}	transverse normal strain (in the z direction)
y_{xy}	in-plane shear strain
y_{xz}, y_{yz}	transverse shear strains
σ_{xx}	in-plane normal stress in the x direction
σ_{yy}	in-plane normal stress in the y direction
σ_{zz}	transverse normal stress (in the z direction)
τ_{xy}	in-plane shear stress
$\tau_{xz},\ \tau_{yz}$	transverse shear stresses
μ	Poisson's ratio
E	Young's modulus of elasticity
G	shear modulus
k_s	transverse shear stress correction factor
$M_{xx},\ M_{yy}$	bending moments
M_{xy}	twisting moments
$Q_x\ \ Q_y$	shear force
D	modulus of flexural rigidity
$N_{xx},\ N_{yy}$	in-plane normal force in the x and y directions respectively
N_{xy}	in-plane shear force
q	transversely applied load distribution
α_m	parameter defined in terms of m, π and a
$\bar{\alpha}_m$	parameter defined in terms of m, π and a
β_n	parameter defined in terms of n, π and b
$\bar{\beta}_n$	parameter defined in terms of n, π and b
N_o	biaxial compressive load
I_{mn}	integrals
N_{yycr}	critical buckling load for uniaxially loaded plate in the y direction
N_{xxcr}	critical buckling load for uniaxially loaded plate in the x direction
N_{ocr}	critical buckling load for biaxially loaded plate in the x and y directions
$m,\ n$	buckling modes
$\bar{N}$	dimensionless critical buckling load parameter for uniaxial buckling in the x direction
KPT	Kirchhoff plate theory
GITM	generalized integral transform method
MPT	Mindlin plate theory
RPT	Reissner plate theory
BCs	boundary conditions
CCCC	clamped on all four edges
SCFrC	clamped along two opposite edges, simply supported along the third edge and free along the fourth edge
SSSS	simply supported on all four edges

HOSNDT	higher order shear and normal deformation plate theory
FG	functionally graded
DFSTM	double finite sine transform method
FoSDPT	first order shear deformation plate theory
TSDPT	trigonometric shear deformation plate theory
PDE	partial differential equation
DEoE	Differential Equations of Equilibrium
GVE	Galerkin Variational Equation

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