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Construction and Utilization of Orthogonal Polynomial for the Fractional Order Integro- Volterra-Fredholm Differential Equations

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Abstract

In this article, a novel orthogonal polynomial is introduced and it is taken as a basis function to solve fractional-order Volterra-Fredholm integro-differential equations (FVFIDEs) using standard and perturbed collocation techniques. We then solve the FVFIDEs by approximating the solution with the constructed orthogonal polynomials, substituting the approximation into the FVFIDEs to generate collocation equations at uniformly spaced interior points, yielding a system of linear algebraic equations. Using Gaussian elimination, we solve this system of equations to find the unknown coefficients back into the assumed solution. To validate the efficiency of the proposed techniques, we present four numerical examples. The results indicate that proposed collocation methods are easy to implement, efficient, and produce results that agree well with existing methods in the literature. This work highlights the robustness and potential of these methods for solving the FVFIDEs with high precision, offering valuable insights into the numerical solutions of the fractional-order Volterra-Fredholm integrodifferential equations that occur in applied mathematics.

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1.0 Introduction

In recent years, there has been a lot of interest in studying fractional integro-differential equations because of their applications in various areas, including physics, engineering, biology, and economics. In particular, it describes well-suited systems with intricate temporal dynamics, such as viscoelastic materials, fluid flow, and population dynamics. The fractional-order derivatives offer a more adaptable framework for simulating memory effects and hereditary characteristics. However, solving these equations analytically is challenging due to their nonlocal nature and the complexity introduced by fractional derivatives. These equations are relevant in various scientific areas, such as non-local phenomena in quantum, diffusion processes, and viscoelastic materials. The fractional derivatives account for time-dependent properties and long-range interactions, making them useful in modeling fluid dynamics, heat conduction, and wave propagation. In engineering sciences, FVFIDEs are employed in control theory to design systems with improved stability and robustness, particularly in areas like automatic control systems, signal processing, and robotics. They also appear in modeling electrical circuits with memory elements such as capacitors and inductors. In biology and medicine, FVFIDEs are used to model processes like tissue perfusion, pharmacokinetics, and population dynamics, where fractional calculus effectively represents the memory and hereditary characteristics of biological systems. In finance FVFIDEs capture long-term dependencies in asset prices and interest rates, making them useful in risk modeling and option pricing [1]-[5].

We consider FVFIDEs of the form:

$$\sum_{i=0}^{n} P_i(x)\phi^i(x) + D^\alpha \phi(x) + \lambda_1 \int_a^{b(x)} K(x,t)\phi(t)\,dt + \lambda_2 \int_a^{b} K(x,t)\phi(t)\,dt = g(x), \qquad (1)$$

Subject to the conditions

$$\phi^{(i)}(a_i) = \psi_i \ ; i = 1,2,3,\ldots,(n-1), \qquad (2)$$

Where $D^\alpha$is the fractional derivatives in Caputo sense and α (real or complex) is the order of the fractional derivative, ψ is a constant, $K(x,t)$ is the kernel function and $g(x)$ is a continuous arbitrary smooth function.

Solving fractional-order integro-differential equations analytically is often challenging due to their nonlocal nature and the complexity introduced by the fractional derivatives. Consequently, numerical methods have become essential tools for obtaining approximate solutions. Several authors have proposed and applied various techniques. For instance, [6] utilized the Adomian decomposition method for the numerical solutions, while, [7] employed a hybrid collocation method. [8] applied the Laplace decomposition method. The homotopy analysis method for higher-order equations was introduced by [8]. Author [9] used analytical methods to solve fractional-order fractional Volterra-Fredholm integro-differential equations and to approximate their solutions. [10] presented approximate solutions of fractional Volterra–Fredholm integro-differential equations by using analytical techniques and [11] used the shifted Chebyshev and least squares approach to numerically solve fractional integro-differential equations. [12] used effective Chebyshev spectral methods for multi-term fractional orders differential equations and suggested a perturbation least-squares method. At the same time [13] employed the spectral-collocation method for the numerical solutions of fractional Fredholm integro-differential equations. [14] used Chebyshev cardinal functions for nonlinear fractional-order Volterra and Fredholm equations, while [15] introduced the Chebyshev approach for solving fractional-order integro-differential equations. [16] presented a novel integral transform, [17] employed the Aboodh transformation method, a numerical method for resolving nonlinear fractional-order Volterra integro-differential equations was presented by [18].

This paper aims to formulate and apply the standard collocation method (SCM) and the perturbed collocation method (PCM). This turns the original problem into a system of algebraic equations by approximating the unknown solution using basis functions and requiring it to satisfy the integro-differential equation at specific collocation points. In order to show how well the approach works in resolving real-world issues represented by fractional-order integro-differential equations, we additionally provide four numerical examples.

2 Definition of fractional Calculus

The following are some fundamental definitions of fractional calculus that are provided in this section: [19]-[22]

2.1 Definition 1.

Fractional calculus involves differentiation and integration for both real numbers and complex values such as $D^{\frac{3}{2}}, D^{\frac{1}{2}}, D^{\frac{7}{2}}$

2.2 Definition 2.

Let $\phi \in L^2([0,T])$, the definition of the Riemann-Liouville fractional derivative (RLFD) is

$$D^\alpha \phi(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \phi(t) dt; \qquad n-1 < \alpha < n \qquad (3)$$

Where n is the smallest value and an integer larger than or equal to α.

2.3 Definition 3.

Let $\phi \in H^n([0,T])$, the Caputo fractional derivative (CFD) of $\phi(x)$ can be defined as

$$D^\beta \phi(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} \phi(t)^{(n)} dt\,; \ \ \beta > 0 \qquad (4)$$

Where $n = [\beta] + 1$.

Hence, we obtain the following properties:

$$\begin{cases} J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{(\alpha+\beta)}, \\ J^\alpha D^\alpha f(x) = g(x); \quad x > 0 \\ D^\alpha C = 0, \quad C \text{ is a constant} \\ D^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{(\gamma-\alpha)}, \end{cases} \qquad (5)$$

Where $[\alpha]$ denotes the integer greater than or equal to α and the smallest value.

3. Construction of orthogonal polynomials

The solutions framework employed in this paper revolves around the construction of orthogonal polynomials, which serve as basis functions for approximating the unknown solution of the FOIDEs. Orthogonal polynomials, known for their desirable properties such as minimizing approximation errors and facilitating numerical computations, were specifically selected for their effectiveness in this context.

3.1 Construction of orthogonal polynomial bases:

The orthogonal polynomials are subjected to standardization for the Legendre polynomial valid in [-1,1] as:

$$Q_n(1) = 1\,, \tag{6}$$

In particular, $Q_0(x) = 1$,

Thus, $Q_n(x)$ is defined in power series form as

$$Q_n(x) = \sum_{i=0}^{n} a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{7}$$

The procedure for constructing our orthogonal polynomials is as follows

$$Q_n(x) = 1 \Rightarrow a_0 + a_1 + a_2 + \cdots + a_n = 1 \tag{8}$$

$$\begin{cases} < Q_n, 1 > \Rightarrow \int_{-1}^{1} w(x) * Q_0(x) * Q_N(x)dx = 0 \\ < Q_n, x > \Rightarrow \int_{-1}^{1} w(x) * Q_1(x) * Q_N(x)dx = 0 \\ < Q_n, x^2 > \Rightarrow \int_{-1}^{1} w(x) * Q_2(x) * Q_N(x)dx = 0 \\ < Q_n, x^3 > \Rightarrow \int_{-1}^{1} w(x) * Q_3(x) * Q_N(x)dx = 0 \\ \vdots \\ < Q_n, x^{n-1} > \Rightarrow \int_{-1}^{1} w(x) * Q_N(x) * Q_N(x)dx = 0 \end{cases} \tag{9}$$

An algebraic linear system of equations with (n+1) unknown constants is obtained. To obtain the orthogonal polynomial, the (n+1) algebraic linear system of equations must be solved.

$$Q_0(x) = 1.$$

The weight function $w(x)$ of Legendre polynomial valid in $[-1,1]$ is 1

$$\begin{cases} Q_0(x) = a_0 \\ Q_1(x) = a_0 + a_1 x \\ Q_2(x) = a_0 + a_1 x + a_2 x^2 \\ \vdots \end{cases} \tag{10}$$

For $Q_1(x)$ from equation (9),

$$< Q_1(x), 1 > \equiv \int_{-1}^{1} w(x) * Q_0(x) * Q_1(x)dx = 0 \tag{11}$$

Recall, $w(x) = 1$ and $Q_0(x) = 1$

$$\begin{cases} < Q_1(x), 1 > \equiv \int_{-1}^{1} Q_1(x)dx = 0 \\ < Q_1(x), 1 > \equiv \int_{-1}^{1} (a_0 + a_1 x)dx = 0 \\ < Q_1(x), 1 > \equiv \left[a_0 x + \frac{a_{1x^2}}{2}\right]_{-1}^{1} = 0 \Rightarrow a_0 = 0 \\ < Q_1(x), 1 > \equiv a_0 = 0 \end{cases} \tag{12}$$

Using the standardization for Legendre polynomial valid in $[-1,1]$, to get:

$$Q_1(1) \equiv a_0 + a_1 = 1 \Rightarrow a_1 = 1$$

That is $$Q_1(x) = x$$

For $Q_2(x)$ from equation (9),

$$< Q_2(x), 1 > \equiv \int_{-1}^{1} w(x) * Q_0(x) * Q_2(x)dx = 0 \quad (13)$$

Recall, $w(x) = 1$ and $Q_0(x) = 1$

$$\begin{cases} < Q_2(x), 1 > \equiv \int_{-1}^{1} Q_2(x)dx = 0 \\ < Q_2(x), 1 > \equiv \int_{-1}^{1} (a_0 + a_1 x + a_2 x^2)dx = 0 \\ < Q_2(x), 1 > \equiv \left[a_0 x + \frac{a_{1x^2}}{2} + \frac{a_{2x^3}}{3}\right]_{-1}^{1} = 0 \\ < Q_1(x), 1 > \equiv 2a_0 + \frac{2a_2}{3} = 0 \end{cases} \quad (14)$$

$$2a_0 + \frac{2a_2}{3} = 0 \quad (15)$$

From (9), we have

$$\begin{cases} < Q_2, x > \Rightarrow \int_{-1}^{1} w(x) * Q_1(x) * Q_2(x)dx = 0 \\ < Q_2, x > \Rightarrow \int_{-1}^{1} 1 * x * (a_0 + a_1 x + a_2 x^2)dx = 0 \\ < Q_2, x > \Rightarrow \int_{-1}^{1} x(a_0 + a_1 x + a_2 x^2)dx = 0 \\ < Q_2, x > \Rightarrow \int_{-1}^{1} (a_0 x + a_1 x^2 + a_2 x^3)dx = 0 \\ a_1 = 0 \end{cases} \quad (16)$$

Using the standardization for Legendre polynomial valid in $[-1,1]$, to get

$$Q_2(1) \equiv a_0 + a_1 + a_2$$

$$\Rightarrow a_0 + a_2 = 1 \quad (17)$$

Solving (15) and (17), to get

$$a_0 = \frac{-1}{2}, \qquad a_2 = \frac{3}{2}$$

From equation (10), we obtain,

$$Q_2(x) = \frac{3}{2}x^2 + \frac{-1}{2}x$$

Then, following the same procedure, the following orthogonal polynomials are obtained:

$$\begin{cases} Q_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \\ Q_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^3 + \frac{3}{8} \\ Q_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^4 + \frac{15}{8}x \\ \vdots \\ \vdots \end{cases} \quad (18)$$

In this paper, the types of problems considered are defined on the interval [0,1]. To address this, we need to convert the orthogonal polynomials from the interval [-1,1] to [0,1]. Therefore, we first carry out the conversion of orthogonal polynomials from the interval [-1,1] to a general interval [a, b], and then specialize to the interval [0,1].

In this case, the linear transformation $x^* = a_1 x + b_1$ (19)

Here, from [-1,1] to [a,b]. Suppose $x = -1$ and $x^* = a,$ $x = 1$ and $x^* = b,$

These resulted in the following system of equations

$$\begin{cases} a = -a_1 + b_1 \\ b = a_1 + b_1 \end{cases} \quad (20)$$

Solving (20) yield

$$\begin{cases} a_1 = \frac{(b-a)}{2} \\ b_1 = \frac{(a+b)}{2} \end{cases} \quad (21)$$

Substituting (21) into (19) we have, $x^* = \frac{(b-a)}{2}x + \frac{(a+b)}{2}$ (22)

And simplifying (22) give $x^* = \frac{2x-(b+a)}{(b-a)}$ (23)

Also, to convert from [-1,1] to [0,1], Here a=0, b=1. Hence (23) becomes

$$x^* = 2x - 1 \quad (24)$$

Substituting $x^* = 2x - 1$ into the constructed orthogonal polynomial (18) yields the orthogonal polynomials on the intervals of [0,1].

$$\begin{cases} Q_0(x) = 1 \\ Q_1(x) = 2x - 1 \\ Q_2(x) = 6x^2 - 6x + 1 \\ Q_3(x) = 20x^3 - 30x^2 + 20x - 1 \\ Q_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1 \\ \vdots \\ \vdots \end{cases} \quad (25)$$

4. Description of proposed methods

In this paper, we propose two types of collocation methods: the SCM and the PCM. These methods are demonstrated on the FVFIDEs (1) and (2) as follows:

4.1 Standard collocation method (SCM) for FVFIDEs

We consider the FVFIDEs of the form:

$$\sum_{i=0}^{n} P_i(x)\phi^i(x) + D^\alpha \phi(x) + \lambda_1 \int_a^{b(x)} K(x,t)\phi(t)\,dt + \lambda_2 \int_a^b K(x,t)\phi(t)\,dt = g(x), \quad (26)$$

Subject to the conditions

$$\phi^{(i)}(a_i) = \psi_i; i = 1,2,3,\dots,(n-1), \quad (27)$$

Equation (26) can be expanded as

$$\begin{cases} P_0(x)\phi(x) + P_1(x)\phi^{/}(x) + P_2(x)\phi^{//}(x) + \cdots + P_n(x)\phi^n(x) + D^\alpha\phi(x) + \\ \lambda_1 \int_a^{b(x)} K(x,t)\phi(t)\,dt + \lambda_2 \int_a^b K(x,t)\phi(t)\,dt = g(x) \end{cases} \quad (28)$$

Let's assume an approximate solution of the form:

$$\phi_N(x) = \sum_{i=0}^{N} c_i Q_i(x) \quad (29)$$

Where N is the computational length and substitute (29) into (28) to get

$$\begin{cases} P_0(x)\phi_N(x) + P_1(x)\phi_N{}^{/}(x) + \cdots + P_n(x)u_N{}^n(x) + D^\alpha\phi_N(x) + \\ \lambda_1 \int_a^{b(x)} K(x,t)\phi_N(t)\,dt + \lambda_2 \int_a^b K(x,t)\phi_N(t)\,dt = g(x) \end{cases} \quad (30)$$

(30) can be expanded as

$$\phi_{N_N}(x) = c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x) \quad (31)$$

Substitute (31) into (30) to get

$$\begin{cases} P_0(x)\big(c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x)\big) + \\ P_1(x)\big(c_0Q_0{}^{/}(x) + c_1Q_1{}^{/}(x) + c_2Q_2{}^{/}(x) + \cdots + c_NQ_N{}^{/}(x)\big) + \\ P_2(x)\big(c_0Q_0{}^{//}(x) + c_1Q_1{}^{//}(x) + c_2Q_2{}^{//}(x) + \cdots + c_NQ_N{}^{//}(x)\big) + \\ \vdots \qquad \vdots \qquad \vdots \\ P_n(x)(c_0Q_0{}^n(x) + c_1Q_1{}^n(x) + c_2Q_2{}^n(x) + \cdots + c_NQ_N{}^n(x)) + \\ D^\alpha\big(c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x)\big) + \\ \lambda_1 \int_a^{b(x)} K(x,t)\big(c_0Q_0(t) + c_1Q_1(t) + c_2Q_2(t) + \cdots + c_NQ_N(t)\big)dt + \\ \lambda_2 \int_a^b K(x,t)\big(c_0Q_0(t) + c_1Q_1(t) + c_2Q_2(t) + \cdots + c_NQ_N(t)\big)\,dt = g(x) \end{cases} \quad (32)$$

Hence, by collecting the like terms in (32), we have

$$\begin{cases} c_0\begin{pmatrix} P_0(x)Q_0(x)+P_1(x){Q_0}^{/}(x)+P_2(x){Q_0}^{//}(x)+\cdots+P_n(x){Q_0}^{(n)}(x)+D^{\alpha}Q_0(x) \\ -\lambda_1\int_a^{b(x)}K(x,t)Q_0(t)-\lambda_2\int_a^{b}K(x,t)Q_0(t) \end{pmatrix} \\ c_1\begin{pmatrix} P_0(x)Q_1(x)+P_1(x){Q_1}^{/}(x)+P_2(x){Q_1}^{//}(x)+\cdots+P_n(x){Q_1}^{(n)}(x)+D^{\alpha}Q_1(x) \\ -\lambda_1\int_a^{b(x)}K(x,t)Q_1(t)-\lambda_2\int_a^{b}K(x,t)Q_1(t) \end{pmatrix} \\ c_2\begin{pmatrix} P_0(x)Q_2(x)+P_1(x){Q_2}^{/}(x)+P_2(x){Q_2}^{//}(x)+\cdots+P_n(x){Q_2}^{(n)}(x)+D^{\alpha}Q_2(x) \\ -\lambda_1\int_a^{b(x)}K(x,t)Q_2(t)-\lambda_2\int_a^{b}K(x,t)Q_2(t) \end{pmatrix} \\ \vdots \\ c_N\begin{pmatrix} P_0(x)Q_N(x)+P_1(x){Q_N}^{/}(x)+P_2(x){Q_N}^{//}(x)+\cdots+P_n(x){Q_N}^{(n)}(x)+D^{\alpha}Q_N(x) \\ -\lambda_1\int_a^{b(x)}K(x,t)Q_N(t)-\lambda_2\int_a^{b}K(x,t)Q_N(t) \end{pmatrix}=g(x) \end{cases} \quad (33)$$

Thus (33) gives (N+1) unknown constants ($c_i; i\geq 0$) to be determined. Consider the initial conditions given in equation (27), we obtain,

$$\begin{cases} \phi_N(a_1)=\psi_0 \Rightarrow c_0Q_0(a_1)+c_1Q_1(a_1)+c_2Q_2(a_1)+\cdots+c_NQ_N(a_1)=\psi_0 \\ {\phi^{/}}_N(a_1)=\psi_1 \Rightarrow c_0{Q^{/}}_0(a_1)+c_1{Q^{/}}_1(a_1)+c_2{Q^{/}}_2(a_1)+\cdots+c_N{Q^{/}}_N(a_1)=\psi_1 \\ {\phi^{//}}_N(a_1)=\psi_2 \Rightarrow c_0{Q^{//}}_0(a_1)+c_1{Q^{//}}_1(a_1)+c_2{Q^{//}}_2(a_1)+\cdots+c_N{Q^{//}}_N(a_1)=\psi_2 \\ \vdots \qquad\qquad \vdots \\ {\phi^{(n-1)}}_N(a_1)=\psi_{(n-1)} \Rightarrow c_0{Q^{(n-1)}}_0(a_1)+c_1{Q^{(n-1)}}_1(a_1)+c_2{Q^{(n-1)}}_2(a_1)+\cdots+c_N{Q^{(n-1)}}_N(a_1)=\psi_{(n-1)} \end{cases} \quad (34)$$

Thus, equation (34) is then collocated at point $x=x_k$ to get,

$$\begin{cases} c_0\begin{pmatrix} P_0(x_k)Q_0(x_k)+P_1(x_k){Q_0}^{/}(x_k)+P_2(x_k){Q_0}^{//}(x)+\cdots+P_n(x_k){Q_0}^{(n)}(x)+D^{\alpha}Q_0(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_0(t)-\lambda_2\int_a^{b}K(x_k,t)Q_0(t) \end{pmatrix} \\ c_1\begin{pmatrix} P_0(x_k)Q_1(x_k)+P_1(x){Q_1}^{/}(x_k)+P_2(x_k){Q_1}^{//}(x)+\cdots+P_n(x_k){Q_1}^{(n)}(x_k)+D^{\alpha}Q_1(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_1(t)-\lambda_2\int_a^{b}K(x_k,t)Q_1(t) \end{pmatrix} \\ c_2\begin{pmatrix} P_0(x_k)Q_2(x_k)+P_1(x_k){Q_2}^{/}(x_k)+P_2(x_k){Q_2}^{//}(x)+\cdots+P_n(x_k){Q_2}^{(n)}(x)+D^{\alpha}Q_2(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_2(t)-\lambda_2\int_a^{b}K(x_k,t)Q_2(t) \end{pmatrix} \\ \vdots \\ c_N\begin{pmatrix} P_0(x)Q_N(x_k)+P_1(x_k){Q_N}^{/}(x)+P_2(x_k){Q_N}^{//}(x_k)+\cdots+P_n(x_k){Q_N}^{(n)}(x_k)+D^{\alpha}Q_N(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_N(t)-\lambda_2\int_a^{b}K(x_k,t)Q_N(t) \end{pmatrix}=g(x_k) \end{cases} \quad (35)$$

Where, $x_k=a+\frac{(b-a)k}{(N-n+2)};\ \ k=1,2,3,\dots,(N-n+1)$,

An algebraic linear system with equations involving (N+1) unknown constants is thus formed by equation (35) as (N-n+1). We get a total of (N+1) algebraic equations with (N+1) unknowns by deriving an extra n equation from (34). To find the approximate answer, these equations are solved to identify the unknown constants, which are then substituted into equation (29)

.

4.2 Perturbed collocation method (PCM) for FVFIDEs

Slightly perturbed (30) from the previous sub-section 4.1 and get,

$$\begin{cases} P_0(x)\phi_N(x) + P_1(x){\phi_N}^{/}(x) + \cdots + P_n(x){u_N}^{n}(x) + D^{\alpha}\phi_N(x) + \\ \lambda_1 \int_a^{b(x)} K(x,t)\phi_N(t)\,dt + \lambda_2 \int_a^{b} K(x,t)\phi_N(t)\,dt = g(x) + \sum_{j=1}^{n} \tau_j Q_{N-n+j}(x) \end{cases} \quad (36)$$

Substitute (31) into (36) to get

$$\begin{cases} P_0(x)\big(c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x)\big) + \\ P_1(x)\big(c_0{Q_0}^{/}(x) + c_1{Q_1}^{/}(x) + c_2{Q_2}^{/}(x) + \cdots + c_N{Q_N}^{/}(x)\big) + \\ P_2(x)\big(c_0{Q_0}^{//}(x) + c_1{Q_1}^{//}(x) + c_2{Q_2}^{//}(x) + \cdots + c_N{Q_N}^{//}(x)\big) + \\ \vdots \qquad \vdots \qquad \vdots \\ P_n(x)(c_0{Q_0}^{n}(x) + c_1{Q_1}^{n}(x) + c_2{Q_2}^{n}(x) + \cdots + c_N{Q_N}^{n}(x)) + \\ D^{\alpha}\big(c_0Q_0(x) + c_1Q_1(x) + c_2Q_2(x) + \cdots + c_NQ_N(x)\big) + \\ \lambda_1 \int_a^{b(x)} K(x,t)\,\big(c_0Q_0(t) + c_1Q_1(t) + c_2Q_2(t) + \cdots + c_NQ_N(t)\big)dt + \\ \lambda_2 \int_a^{b} K(x,t)\big(c_0Q_0(t) + c_1Q_1(t) + c_2Q_2(t) + \cdots + c_NQ_N(t)\big)\,dt = g(x) + \sum_{j=1}^{n} \tau_j Q_{N-n+j}(x) \end{cases} \quad (37)$$

Hence, collecting like terms in (37), we have

$$\begin{cases} c_0\begin{pmatrix} P_0(x)Q_0(x) + P_1(x){Q_0}^{/}(x) + P_2(x){Q_0}^{//}(x) + \cdots + P_n(x){Q_0}^{(n)}(x) + D^{\alpha}Q_0(x) \\ -\lambda_1 \int_a^{b(x)} K(x,t)Q_0(t) - \lambda_2 \int_a^{b} K(x,t)Q_0(t) \end{pmatrix} \\ c_1\begin{pmatrix} P_0(x)Q_1(x) + P_1(x){Q_1}^{/}(x) + P_2(x){Q_1}^{//}(x) + \cdots + P_n(x){Q_1}^{(n)}(x) + D^{\alpha}Q_1(x) \\ -\lambda_1 \int_a^{b(x)} K(x,t)Q_1(t) - \lambda_2 \int_a^{b} K(x,t)Q_1(t) \end{pmatrix} \\ c_2\begin{pmatrix} P_0(x)Q_2(x) + P_1(x){Q_2}^{/}(x) + P_2(x){Q_2}^{//}(x) + \cdots + P_n(x){Q_2}^{(n)}(x) + D^{\alpha}Q_2(x) \\ -\lambda_1 \int_a^{b(x)} K(x,t)Q_2(t) - \lambda_2 \int_a^{b} K(x,t)Q_2(t) \end{pmatrix} \\ \vdots \\ c_N\begin{pmatrix} P_0(x)Q_N(x) + P_1(x){Q_N}^{/}(x) + P_2(x){Q_N}^{//}(x) + \cdots + P_n(x){Q_N}^{(n)}(x) + D^{\alpha}Q_N(x) \\ -\lambda_1 \int_a^{b(x)} K(x,t)Q_N(t) - \lambda_2 \int_a^{b} K(x,t)Q_N(t) \end{pmatrix} = \\ g(x) + \sum_{j=1}^{n} \tau_j Q_{N-n+j}(x) \end{cases} \quad (38)$$

Thus (38) gives (N+n+1) unknown constants ($c_i; i \geq 0$) to be determined. We considered the initial conditions given in equation (27), and we obtained,

$$\begin{cases} \phi_N(a_1) = \psi_0 \Rightarrow c_0Q_0(a_1) + c_1Q_1(a_1) + c_2Q_2(a_1) + \cdots + c_NQ_N(a_1) = \psi_0 \\ {\phi^{/}}_N(a_1) = \psi_1 \Rightarrow c_0{Q^{/}}_0(a_1) + c_1{Q^{/}}_1(a_1) + c_2{Q^{/}}_2(a_1) + \cdots + c_N{Q^{/}}_N(a_1) = \psi_1 \\ {\phi^{//}}_N(a_1) = \psi_2 \Rightarrow c_0{Q^{//}}_0(a_1) + c_1{Q^{//}}_1(a_1) + c_2{Q^{//}}_2(a_1) + \cdots + c_N{Q^{//}}_N(a_1) = \psi_2 \\ \vdots \qquad \vdots \\ {\phi^{(n-1)}}_N(a_1) = \psi_{(n-1)} \Rightarrow c_0{Q^{(n-1)}}_0(a_1) + c_1{Q^{(n-1)}}_1(a_1) + c_2{Q^{(n-1)}}_2(a_1) + \cdots + c_N{Q^{(n-1)}}_N(a_1) = \psi_{(n-1)} \end{cases} \quad (39)$$

Thus, equation (38) is then collocated at point $x = x_k$ to get,

$$\begin{cases} c_0\begin{pmatrix} P_0(x_k)Q_0(x_k)+P_1(x_k)Q_0{}^{/}(x_k)+P_2(x_k)Q_0{}^{//}(x)+\cdots+P_n(x_k)Q_0{}^{(n)}(x)+D^{\alpha}Q_0(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_0(t)-\lambda_2\int_a^{b}K(x_k,t)Q_0(t) \end{pmatrix} \\ c_1\begin{pmatrix} P_0(x_k)Q_1(x_k)+P_1(x)Q_1{}^{/}(x_k)+P_2(x_k)Q_1{}^{//}(x)+\cdots+P_n(x_k)Q_1{}^{(n)}(x_k)+D^{\alpha}Q_1(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_1(t)-\lambda_2\int_a^{b}K(x_k,t)Q_1(t) \end{pmatrix} \\ c_2\begin{pmatrix} P_0(x_k)Q_2(x_k)+P_1(x_k)Q_2{}^{/}(x_k)+P_2(x_k)Q_2{}^{//}(x)+\cdots+P_n(x_k)Q_2{}^{(n)}(x)+D^{\alpha}Q_2(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_2(t)-\lambda_2\int_a^{b}K(x_k,t)Q_2(t) \end{pmatrix} \\ \vdots \\ c_N\begin{pmatrix} P_0(x)Q_N(x_k)+P_1(x_k)Q_N{}^{/}(x)+P_2(x_k)Q_N{}^{//}(x_k)+\cdots+P_n(x_k)Q_N{}^{(n)}(x_k)+D^{\alpha}Q_N(x_k) \\ -\lambda_1\int_a^{b(x)}K(x_k,t)Q_N(t)-\lambda_2\int_a^{b}K(x_k,t)Q_N(t) \end{pmatrix}= \\ g(x_k)+\sum_{j=1}^{n}\tau_j Q_{N-n+j}(x_k) \end{cases} \quad (40)$$

where $x_k = a + \frac{(b-a)k}{(N+2)};\ \ k = 1,2,3,\dots,(N+1)$.

An algebraic linear system with equations involving (N+3) unknown constants is thus formed by equation (40). We get a total of (N+3) algebraic equations with (N+3) unknowns by deriving an extra n equation from (39). The unknown constants are found by solving these equations, and the approximate answers are then obtained by substituting them into equation (29).

4.3 Convergence and Error analysis

This section discusses the convergence and error analysis of the proposed SCM and PCM applied to fractional-order integral-Volterra-Fredholm differential equations. The analysis highlights their performance and accuracy in solving these complex equations, evaluating their effectiveness in terms of convergence rates and error bounds.

4.3.1 Consistency: Define the residual $R_N(x)$ as:

$$\begin{cases} R_N(x)=\sum_{i=0}^{n}P_i(x)\phi^i(x)+D^{\alpha}\,\phi(x)+\lambda_1\int_a^{b(x)}K(x,t)\phi(t)\,dt+ \\ \lambda_2\int_a^{b}K(x,t)\phi(t)\,dt-g(x) \end{cases} \quad (41)$$

If $\phi(x)$ is the exact solution, then.

$$\begin{cases} R_N(x)=\sum_{i=0}^{n}P_i(x)\phi^i(x)+D^{\alpha}\,\phi(x)+\lambda_1\int_a^{b(x)}K(x,t)\phi(t)\,dt+ \\ \lambda_2\int_a^{b}K(x,t)\phi(t)\,dt-g(x)=0 \end{cases} \quad (42)$$

For $\phi_N(x)$ to converge to $\phi(x)$ as $N\to\infty$
$R_N(x)$ converge to 0 uniformly on $[a,b]$: $\|R_N(x)\|\to 0$ as $N\to\infty$

4.3.2 Stability: Stability is determined by the properties of the matrix of the linear system obtained from collocation

$$Ac=b \quad (43)$$

Where A is the matrix formed by evaluating the derivatives of basis functions and the fractional derivative term at the collocation points, c is the vector of coefficients $\{c_k\}$, and b is the vector of $f(x_k)$. Stability is ensured if the condition number of A remains bounded as $N\to\infty$.

4.3.3 Uniqueness: The linear system $Ac=b$ has a unique solution if A is invertible. The invertibility of A is guaranteed if the basis functions $Q_i(x)$ and the collocation points x_k are chosen appropriately.

4.3.4 Error Analysis:

The error $E_N(x) = \phi_N(x) - \phi_i(x) \qquad i \geq 0$ (44)

Depends on the smoothness of the exact solution $\phi(x)$ and the completeness of the basis functions

$$\|R_N(x)\| = \left\| \sum_{i=0}^{n} P_i(x)\phi^i(x) + D^\alpha \phi(x) + \lambda_1 \int_a^{b(x)} K(x,t)\phi(t)\,dt + \lambda_2 \int_a^b K(x,t)\phi(t)\,dt - g(x) \right\| \quad (45)$$

The error is bounded by

$$\|E_N(x)\| \leq C max_{x\in[a,b]} \|R_N(x)\| \quad (46)$$

Where C is a constant dependent on the matrix properties and the problem domain.

5. Computational Application

We present four numerical examples to demonstrate the effectiveness of the proposed techniques as follows:

Example 1. Consider the FVFIDE of the form [6]

$$D^\alpha \phi(x) = \frac{x^\alpha}{\Gamma(1.5)} - \frac{x^2}{2} - \frac{x^2 e^x}{3}\phi(x) + \int_0^x e^x t\phi(t)\,dt + \int_0^1 x^2\phi(t)\,dt, \quad (47)$$

initial condition: $\phi(0) = 0$ (48)

The exact solution of equations (47)and (48) is $\phi(x) = x$ (49)

for $\alpha = 0.5$

Example 2. Consider the FVFIDE of the form [23]

$$D^\alpha \phi(x) + \phi(x)^{//} = g(x) - 2\int_0^x (x-t)\,dt + \int_0^1 (x^2 - t)\phi(t)\,dt\,, \quad (49)$$

boundary conditions: $\phi(0) = 0\,, \quad \phi(1) = 0$ (50)

where $g(x) = -\frac{1}{30} - 6x + \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{6}{16}$

The exact solution of equations (49) and (50) is $\phi(x) = x^3(x-1)$ (51)

for $\alpha = 1.0$

Example 3. Consider the FVFIDE of the form [24]

$$D^\alpha \phi(x) = g(x) + \frac{1}{2}\int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}}\,dt + \frac{1}{3}\int_0^1 (x-t)\phi(t)\,dt \quad (52)$$

initial condition: $\phi(0) = 0$ (53)

where $g(x) = \frac{\Gamma(3)x^{1.75}}{\Gamma(2.75)} + \frac{\Gamma(4)x^{2.75}}{\Gamma(3.75)} - \frac{\sqrt{\pi}\Gamma(3)x^{\frac{5}{2}}}{2\Gamma\left(\frac{7}{2}\right)} - \frac{\sqrt{\pi}\Gamma(4)x^{\frac{7}{2}}}{2\Gamma\left(\frac{9}{2}\right)} - \frac{7x}{36} + \frac{3}{20}$

The exact solution of equations (52) and (53) is $\phi(x) = x^2 + x^3$ (54)

for $\alpha = 0.25$

Example 4. Consider the FVFIDE of the form [24]

$$D^\alpha \phi(x) = g(x) + \frac{1}{4}\int_0^x \frac{\phi(t)}{(x-t)^{\frac{1}{2}}}\,dt + \frac{1}{7}\int_0^1 e^{x+t}\phi(t)\,dt \quad (55)$$

initial condition: $\phi(0) = 0$ (56)

where $g(x) = \frac{\Gamma(3)x^{1.85}}{\Gamma(2.85)} - \frac{\Gamma(2)x^{0.85}}{\Gamma(1.85)} - \frac{\sqrt{\pi}\Gamma(3)x^{\frac{5}{2}}}{4\Gamma\left(\frac{7}{2}\right)} + \frac{\sqrt{\pi}\Gamma(2)x^{\frac{3}{2}}}{4\Gamma\left(\frac{5}{2}\right)} - \frac{e^{x+1} - 3e^x}{7}$

The exact solution of equations (55) and (56) is $\phi(x) = x(x-1)$ (57)

for $\alpha = 0.15$

5.1 Numerical results and graphs representation

We applied the proposed methods (SCM and PCM) for the different computational lengths and the approximate results from the SCM and PCM are compared with the method available in the literature and presented as follow:

Example 1.

$$\begin{cases} \phi(x_{\alpha=0.5})_{SCM} \cong 0.9999999x - 2.6395136x^2e - 9 + 5.3492991x^3e - 9 - 2.5222864x^4e - 9 \\ \phi(x_{\alpha=0.5})_{PCM} \cong 1.00000002x - 9.65002896x^2e - 9 + 1.486998x^3e - 8 - 6.9573838x^4e - 9 \end{cases} \quad (58)$$

Table 1. Numerical solutions for Example 1 at the $\alpha = 0.5$ and $N = 4$

x	Exact solution	SCM	PCM	[6]
0.0	0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.1	0.100000000000	0.100000000000	0.100000000000	0.100000000000
0.2	0.200000000000	0.200000000000	0.200000000000	0.200000000000
0.3	0.300000000000	0.300000000000	0.300000000000	0.300000000000
0.4	0.400000000000	0.400000000000	0.400000000000	0.400000000000
0.5	0.500000000000	0.500000000000	0.500000000000	0.500000000000
0.6	0.600000000000	0.600000000000	0.600000000000	0.600000000000
0.7	0.700000000000	0.700000000000	0.700000000000	0.700000000000
0.8	0.800000000000	0.800000000000	0.800000000000	0.800000000000
0.9	0.900000000000	0.900000000000	0.900000000000	0.900000000000
1.0	1.000000000000	1.000000000000	1.000000000000	1.000000000000

Example 2.

$$\begin{cases} \phi(x_{\alpha=1.0})_{SCM} \cong 1.000e - 11 - 3.8xe - 10 - 3.00x^2e - 10 - 0.999001x^3 + 0.99999x^4 \\ \phi(x_{\alpha=1.0})_{PCM} \cong -1.38xe - 9 + 7.000x^2e - 9 - 1.0000x^3 + 1.000000004x^4 \end{cases} \quad (59)$$

Table 2. Absolute errors for Example 2 at the $\alpha = 1.0$ and $N = 4$

x	SCM	PCM	[23]
0.1	3.02E-11	3.02E-11	3.395E-3
0.2	7.30E-11	7.30E-11	6.465E-3
0.3	1.00E-10	1.00E-10	8.649E-3
0.4	1.50E-10	1.50E-10	9.362E-3
0.5	1.80E-10	1.80E-10	8.227E-3
0.6	2.30E-10	2.30E-10	5.285E-3
0.7	2.10E-10	2.10E-10	1.175E-3
0.8	3.10E-10	3.10E-10	2.719E-3
0.9	2.70E-10	2.70E-10	4.167E-3

Example 3.

$$\begin{cases} \phi(x_{\alpha=0.25})_{SCM} \cong -5.00xe - 9 + 1.0000000x^2 + 0.999999999x^3 + 4.694697x^4e - 8 \\ \phi(x_{\alpha=0.25})_{PCM} \cong 1.000000xe - 9 + 1.000000x^2 + 0.9999999x^3 + 1.649235x^4e - 8 \end{cases} \quad (60)$$

Table 3. Numerical solutions for Example 3 at the $\alpha = 0.25$ and $N = 4$

x	Exact solution	SCM	PCM
0.0	0.000000000000	0.00000000000	0.00000000000
0.1	0.011000000000	0.01099999975	0.01099999975
0.2	0.048000000000	0.04799999977	0.04799999977
0.3	0.117000000000	0.11699999980	0.11699999980
0.4	0.224000000000	0.22399999980	0.22399999980
0.5	0.375000000000	0.37499999970	0.37499999970
0.6	0.57600000000	0.5759999996	0.5759999996
0.7	0.83300000000	0.8329999997	0.8329999997
0.8	1.15200000000	1.1520000000	1.1520000000
0.9	1.53900000000	1.5390000020	1.5390000020
1.0	2.000000000	2.0000000050	2.0000000050

Example 4.

$$\begin{cases}\phi(x_{\alpha=0.15})_{SCM} \cong 1.0000e-10-1.0000x+0.999999x^2+1.442766x^3e-9-6.1588616x^4e-8 \\ \phi(x_{\alpha=0.15})_{PCM} \cong -1.000e-10-1.0000022x+1.00000x^2-1.558914x^3e-7+6.96522x^4e-8\end{cases} \quad (61$$

Table 4. Numerical solutions for Example 4 at the $\alpha = 0.15$ and $N = 4$

x	Exact solution	SCM	PCM	[24]
0.0	0.00000000000	0.00000000000	0.00000000000	0.00000000000
1/8	-0.1093750000	-0.1093749999	-0.1093750015	-0.1088000000
2/8	-0.1875000000	-0.1874999999	-0.1875000011	-0.1861000000
3/8	-0.2343750000	-0.2343750000	-0.2343750002	-0.2320000000
4/8	-0.2500000000	-0.2500000001	-0.2499999997	-0.2497000000
5/8	-0.2343750000	-0.2343750000	-0.2343749999	-0.2332000000
6/8	-0.1875000000	-0.1875000001	-0.1875000007	-0.1862000000
7/8	-0.1093750000	-0.1093750000	-0.1093750017	-0.1081000000

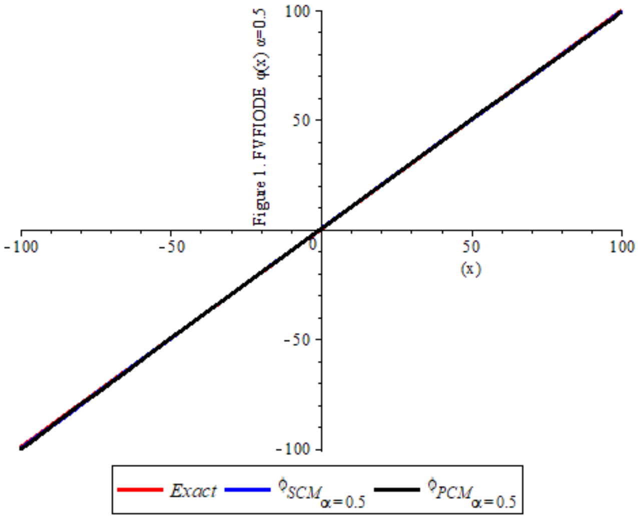

Figure. 1. Depict numerical solutions for example 1

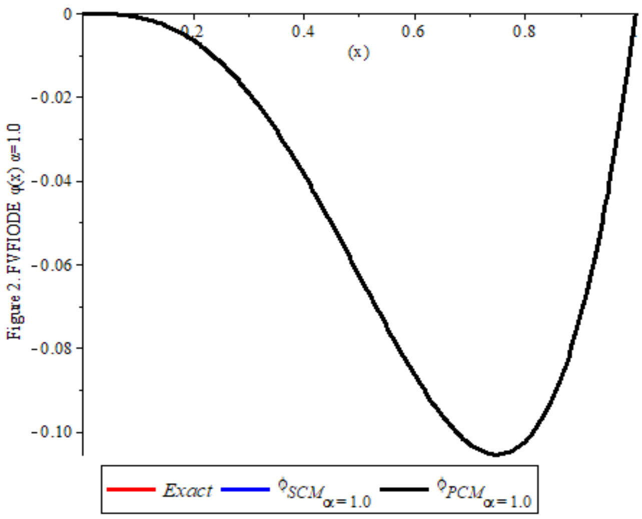

Figure. 2. Depict numerical solutions for example 2

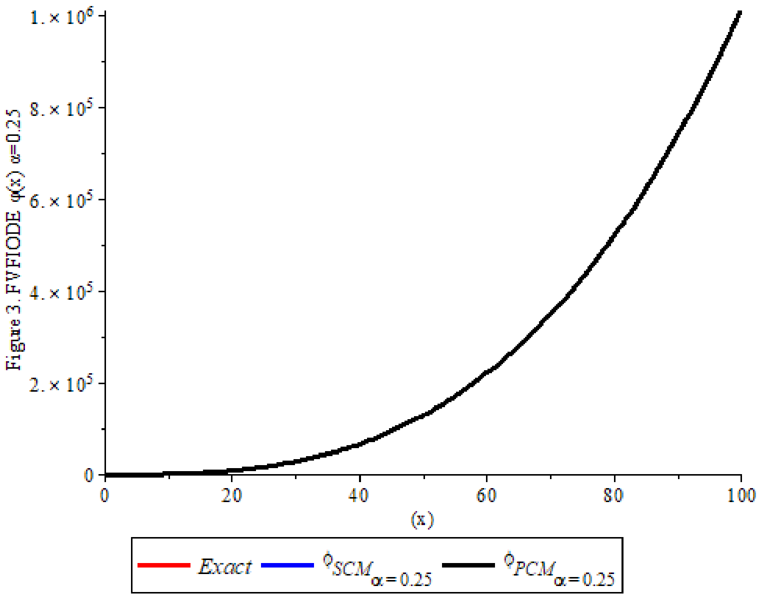

Figure 3. Depict numerical solutions for example 3

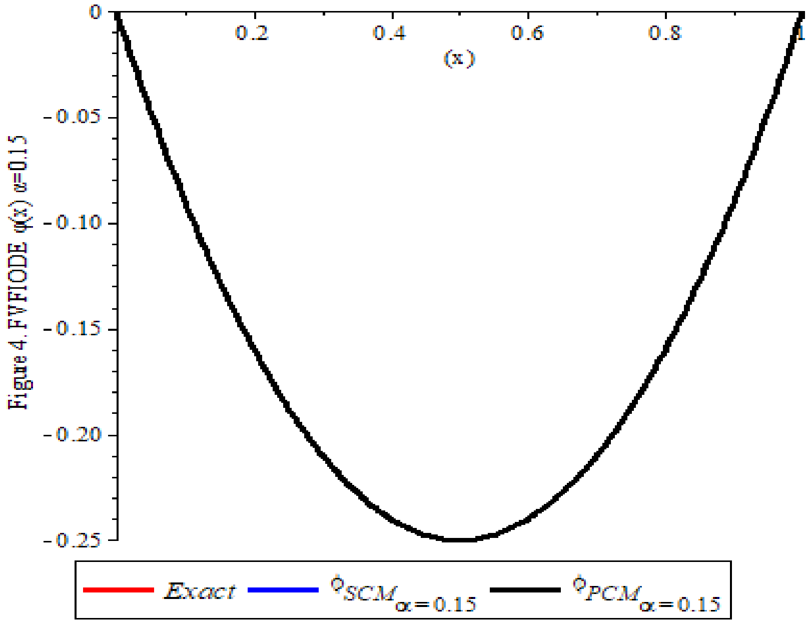

Figure 4. Depict numerical solutions for example 4

5.2 Discussion

FVFIDEs have numerous applications in disciplines including physics, engineering, and biology. The significance of FVFIDEs is an important subject of study and their fractional derivatives and integrals, which stand for non-local interactions and memory effects, these equations combine aspects of the Volterra and Fredholm equations which poses challenges of obtaining the analytical solutions or unattainable in most situations. Therefore, we considered four examples from the available literature and the results demonstrated a good agreement with the exact solutions and the existing methods in which the results are presented in Tables 1,2,3, and 4 and Figures1,2,3 and 4.

6.0 Conclusion

Solving FVFIDEs is a complex yet essential task due to their extensive applications in modelling systems with memory, nonlocal interactions, and hereditary properties. The two proposed techniques (SCM and PCM) have proven effective for FVFIDEs considered in this paper. The results presented in the tables and graphs demonstrate the efficacy of these methods in solving FVFIDEs, yielding solutions with remarkable accuracy. The comparison with existing methods available in the literature, the SCM and PCM tend to be less computational steps length N for high accuracy and convergence (see Tables 1, 2, 3, and 4). However, PCM's reduction in the number of collocation points can make it more efficient than SCM and other traditional methods, especially for problems where high accuracy with fewer points is desired. Although both methods exhibit comparable accuracy, making it challenging to identify a single superior approach, we conclude that both techniques are effective and viable options for solving similar problems in applied sciences and computational engineering. All simplifications, computations, and plots were performed using the Maple 18 software package (see appendix).

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Appendix

Step 1 Example 3

```
restart : with(plots); Digits := 15 : SCM :
```

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, setoptions3d, spacecurve, sparsematrixplot, surfdata, textplot, textplot3d, tubeplot*]

$q_0 := 1 :$

$q_1 := 2x - 1 :$

$q_2 := 6x^2 - 6x + 1 :$

$q_3 := 20x^3 - 30x^2 + 12x - 1 :$

$q_4 := 70x^4 - 140x^3 + 90x^2 - 20x + 1 :$

$q_5 := 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1 :$

$q_6 := 924 \cdot x^6 - 2772 \cdot x^5 + 3150 \cdot x^4 - 1680 \cdot x^3 + 420 \cdot x^2 - 42 \cdot x - 1 :$

$\phi_4 := expand(c_0 \cdot q_0 + c_1 \cdot q_1 + c_2 \cdot q_2 + c_3 \cdot q_3 + c_4 \cdot q_4 + c_5 \cdot q_5 + c_6 \cdot q_6) \ :$

$$\phi_4 := 924 c_6 x^6 + 252 c_5 x^5 - 2772 c_6 x^5 + 70 c_4 x^4 - 630 c_5 x^4 + 3150 c_6 x^4 + 20 c_3 x^3 - 140 c_4 x^3 + 560 c_5 x^3 - 1680 c_6 x^3 + 6 c_2 x^2 - 30 c_3 x^2 + 90 c_4 x^2 - 210 c_5 x^2 + 420 c_6 x^2 + 2 c_1 x - 6 c_2 x + 12 c_3 x - 20 c_4 x + 30 c_5 x - 42 c_6 x + c_0 - c_1 + c_2 - c_3 + c_4 - c_5 - c_6$$

Step 2

$\phi := \dfrac{\mathrm{GAMMA}(n+1)}{\mathrm{GAMMA}(n-k+1)} \cdot x^{(n-k)} :$

$D_0 := evalf(subs(\{n = 0, k = 0.25\}, \phi)) :$

$D_1 := evalf(subs(\{n = 1, k = 0.25\}, \phi)) :$

$D_2 := evalf(subs(\{n = 2, k = 0.25\}, \phi)) :$

$D_3 := evalf(subs(\{n = 3, k = 0.25\}, \phi)) :$

$D_4 := evalf(subs(\{n = 4, k = 0.25\}, \phi)) :$

$D_5 := evalf(subs(\{n=5, k=0.25\}, \phi)) : D_6 := evalf(subs(\{n=6, k=0.25\}, \phi)) :$

$\phi_5 := subs(x=t, \phi_4) :$

$$\phi_6 := expand(924\,c_6 \cdot D_6 + (252\,c_5 - 2772\,c_6)\cdot D_5 + (70\,c_4 - 630\,c_5 + 3150\,c_6)\cdot D_4 + (20\,c_3 - 140\,c_4 + 560\,c_5 - 1680\,c_6)\cdot D_3 + (6\,c_2 - 30\,c_3 + 90\,c_4 - 210\,c_5 + 420\,c_6)\cdot D_2 + (2\,c_1 - 6\,c_2 + 12\,c_3 - 20\,c_4 + 30\,c_5 - 42\,c_6)\cdot D_1 + (c_0 - c_1 + c_2 - c_3 + c_4 - c_5 - c_6)\cdot D_0);$$

$$1468.574369\,c_6\,x^{5.75} + 383.8319375\,x^{4.75}\,c_5 - 4222.151312\,x^{4.75}\,c_6 + 101.2889835\,x^{3.75}\,c_4 - 911.6008511\,x^{3.75}\,c_5 + 4558.004256\,x^{3.75}\,c_6 + 27.13097772\,x^{2.75}\,c_3 - 189.9168440\,x^{2.75}\,c_4 + 759.6673762\,x^{2.75}\,c_5 - 2279.002128\,x^{2.75}\,c_6 + 7.461018870\,x^{1.75}\,c_2 - 37.30509435\,x^{1.75}\,c_3 + 111.9152830\,x^{1.75}\,c_4 - 261.1356604\,x^{1.75}\,c_5 + 522.2713209\,x^{1.75}\,c_6 + 2.176130504\,x^{0.75}\,c_1 - 6.528391512\,x^{0.75}\,c_2 + 13.05678302\,x^{0.75}\,c_3 - 21.76130504\,x^{0.75}\,c_4 + 32.64195756\,x^{0.75}\,c_5 - 45.69874058\,x^{0.75}\,c_6 + \frac{0.8160489394\,c_0}{x^{0.25}} - \frac{0.8160489394\,c_1}{x^{0.25}} + \frac{0.8160489394\,c_2}{x^{0.25}} - \frac{0.8160489394\,c_3}{x^{0.25}} + \frac{0.8160489394\,c_4}{x^{0.25}} - \frac{0.8160489394\,c_5}{x^{0.25}} - \frac{0.8160489394\,c_6}{x^{0.25}}$$

Step 3

$$\phi_7 := evalf\left(\phi_6 = \frac{GAMMA(3)\cdot(x)^{1.75}}{GAMMA(2.75)} + \frac{GAMMA(4)\cdot(x)^{2.75}}{GAMMA(3.75)} - \frac{\sqrt{\pi}\cdot x^{\frac{5}{2}}\cdot\Gamma(3)}{2\cdot GAMMA\left(\frac{7}{2}\right)} - \frac{\sqrt{\pi}\cdot x^{\frac{7}{2}}\cdot\Gamma(4)}{2\cdot GAMMA\left(\frac{9}{2}\right)} - \frac{7\cdot x}{36} + \frac{3}{20} + \frac{1}{2}\int_0^x \frac{1}{(x-t)^{\frac{1}{2}}}\cdot\phi_5\, dt + \frac{1}{3}\cdot\int_0^1 (x-t)\cdot\phi_5\, dt\right);$$

>

$$\phi_7 := 1468.574369 c_6 x^{5.75} + 383.8319375 x^{4.75} c_5 - 4222.151312 x^{4.75} c_6 + 101.2889835 x^{3.75} c_4 - 911.6008511 x^{3.75} c_5 + 4558.004256 x^{3.75} c_6 + 27.13097772 x^{2.75} c_3 - 189.9168440 x^{2.75} c_4 + 759.6673762 x^{2.75} c_5 - 2279.002128 x^{2.75} c_6 + 7.461018870 x^{1.75} c_2 - 37.30509435 x^{1.75} c_3 + 111.9152830 x^{1.75} c_4 - 261.1356604 x^{1.75} c_5 + 522.2713209 x^{1.75} c_6 + 2.176130504 x^{0.75} c_1 - 6.528391512 x^{0.75} c_2 + 13.05678302 x^{0.75} c_3 - 21.76130504 x^{0.75} c_4 + 32.64195756 x^{0.75} c_5 - 45.69874058 x^{0.75} c_6 + \frac{0.8160489394 c_0}{x^{0.25}} - \frac{0.8160489394 c_1}{x^{0.25}} + \frac{0.8160489394 c_2}{x^{0.25}} - \frac{0.8160489394 c_3}{x^{0.25}} + \frac{0.8160489394 c_4}{x^{0.25}} - \frac{0.8160489394 c_5}{x^{0.25}} - \frac{0.8160489394 c_6}{x^{0.25}} = -0.1944444444 x + 0.1500000000 + 0.3333333333 x (c_0 - 1. c_1 + c_2 - 1. c_3 + c_4 - 1. c_5 - 1. c_6) + 0.1666666667 x (2. c_1 - 6. c_2 + 12. c_3 - 20. c_4 + 30. c_5 - 42. c_6) + 0.1111111111 x (6. c_2 - 30. c_3 + 90. c_4 - 210. c_5 + 420. c_6) + 0.08333333333 x (20. c_3 - 140. c_4 + 560. c_5 - 1680. c_6) + 0.06666666667 x (70. c_4 - 630. c_5 + 3150. c_6) + 0.05555555556 x (252. c_5 - 2772. c_6) + 315.0769231 x^{13/2} c_6 + 93.09090909 x^{11/2} c_5 - 1024. x^{11/2} c_6 + 28.44444444 x^{9/2} c_4 - 256. x^{9/2} c_5 + 9.142857143 x^{7/2} c_3 - 64. x^{7/2} c_4 + 3.200000000 x^{5/2} c_2 - 16. x^{5/2} c_3 + 224. x^{5/2} c_6 + 1280. x^{9/2} c_6 + 256. x^{7/2} c_5 - 768. x^{7/2} c_6 + 48. x^{5/2} c_4 - 112. x^{5/2} c_5 - 4. x^{3/2} c_2 + 8. x^{3/2} c_3 - 13.33333333 x^{3/2} c_4 + 20. x^{3/2} c_5 - 28. x^{3/2} c_6 + 1.333333333 x^{3/2} c_1 + \sqrt{x} c_2 - 1. \sqrt{x} c_3 + \sqrt{x} c_4 - 1. \sqrt{x} c_5 - 1. \sqrt{x} c_6 + \sqrt{x} c_0 - 1. \sqrt{x} c_1 + 44. c_6 x - 0.4571428571 x^{7/2} - 0.5333333333 x^{5/2} + 1.356548886 x^{2.75} + 1.243503145 x^{1.75} + 0.3333333333 c_6 - 0.1666666667 c_0 - 0.05555555556 c_1$$

$\phi_8 := evalf(subs(\{x = 0\}, \phi_4) = 0)$:

$\phi_9 := evalf\left(subs\left(x = \frac{1}{7}, \phi_7\right)\right)$:

$\phi_{10} := evalf\left(subs\left(x = \frac{2}{7}, \phi_7\right)\right)$:

$\phi_{11} := evalf\left(subs\left(x = \frac{3}{7}, \phi_7\right)\right)$:

$\phi_{12} := evalf\left(subs\left(x = \frac{4}{7}, \phi_7\right)\right)$:

$\phi_{13} := evalf\left(subs\left(x = \frac{5}{7}, \phi_7\right)\right)$:

$\phi_{14} := evalf\left(subs\left(x = \frac{6}{7}, \phi_7\right)\right)$:

$\phi_{17} := solve(\{\phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14}\}, [c_0, c_1, c_2, c_3, c_4, c_5, c_6])$

$\phi_{17} := [[c_0 = 0.5833333366, c_1 = 0.9500000020, c_2 = 0.4166666694, c_3 = 0.05000000342, c_4 = 3.281198427\ 10^{-9}, c_5 = 2.270966903\ 10^{-9}, c_6 = 1.658614811\ 10^{-9}]]$

$\phi_{18} := evalf(subs(\{c_0 = 0.5833333366, c_1 = 0.9500000020, c_2 = 0.4166666694, c_3 = 0.05000000342, c_4 = 3.281198427\ 10^{-9}, c_5 = 2.270966903\ 10^{-9}, c_6 = 1.658614811\ 10^{-9}\}, \phi_4))$

$\phi_{18} := 0.000001532560085x^6 - 0.000004025396596x^5 + 0.000004023611396x^4 + 0.9999980939x^3 + 1.000000428x^2 - 3.8\ 10^{-8}x - 1.\ 10^{-10}$

$SCM := evalf(subs(\{c_0 = 0.5833333366, c_1 = 0.9500000020, c_2 = 0.4166666694, c_3 = 0.05000000342, c_4 = 3.281198427\ 10^{-9}, c_5 = 2.270966903\ 10^{-9}, c_6 = 1.658614811\ 10^{-9}\}, \phi_4))$

>

$SCM := 0.000001532560085x^6 - 0.000004025396596x^5 + 0.000004023611396x^4 + 0.9999980939x^3 + 1.000000428x^2 - 3.8\ 10^{-8}x - 1.\ 10^{-10}$

$Exact := x^3 + x^2$

Step 4

$\phi_{exact} := x^2 + x^3;$

$x^3 + x^2$

$\phi_{SCM}[0.25] := 4.694697111\ 10^{-8}x^4 + 0.9999999321x^3 + 1.000000031x^2 - 5.\ 10^{-9}x;$

$4.694697111\ 10^{-8}x^4 + 0.9999999321x^3 + 1.000000031x^2 - 5.000000000\ 10^{-9}x$

$\phi_{PCM}[0.25] := 1.649235085\ 10^{-8}x^4 + 0.9999999830x^3 + 1.000000002x^2 + 1.\ 10^{-9}x;$

$1.649235085\ 10^{-8}x^4 + 0.9999999830x^3 + 1.000000002x^2 + 1.000000000\ 10^{-9}x$

$plot([\phi_{exact}, \phi_{SCM}[0.25], \phi_{PCM}[0.25]], x = 0..100, legend = [Exact, \phi_{SCM}[\alpha = 0.25], \phi_{PCM}[\alpha = 0.25]], color = [red, blue, black],$ **labels** = ["(x)"," Figure 3. FVFIODE ϕ(x) α = 0.25"], **labeldirections** = [**HORIZONTAL**, **VERTICAL**])

restart : **with**(**plots**) : **Digits** := **5** : $\alpha := 0.25$:

$\phi_{exact} := x^2 + x^3$:

$\phi_{scm}[0.25] := 4.694697111\ 10^{-8}x^4 + 0.9999999321x^3 + 1.000000031x^2 - 5.\ 10^{-9}x$:

$\phi_{pcm}[0.25] := 1.649235085\ 10^{-8}x^4 + 0.9999999830x^3 + 1.000000002x^2 + 1.\ 10^{-9}x$:

for *n* **from** 0 **by** 0.1 **to** 1 **do**

$E[n] := evalf(eval(\phi_{exact}, x = n))$:

$\phi[n] := evalf(eval(\phi_{scm}[0.25], x = n))$:

$\phi[n] := evalf(eval(\phi_{scm}[0.25], x = n));$

$Error[n] := \text{abs}(E[n] - \phi[n])$:

$Error[n] := \text{abs}(E[n] - \phi[n])$:

end do;

0.
0.
0.
0.
0.
0.011
0.011000
0.011000
0.
0.
0.048
0.048000
0.048000
0.
0.
0.117
0.11700
0.11700
0.
0.
0.224
0.22400
0.22400
0.
0.
0.375
0.37500
0.37500
0.
0.
0.576
0.57600
0.57600
0.
0.
0.833
0.83300
0.83300
0.
0.
1.152
1.1520
1.1520
0.
0.
1.539
1.5390
1.5390
0.
0.
2.000
2.0000
2.0000
0.