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Analytic Approximation of Non-Periodic Dynamical System by Perturbation Method

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Abstract

In this study, perturbation method is applied to obtain the solution of dynamical system which does not involve periodic solution. Since perturbation method presents results in series approximation form, accuracies with retention of different number of terms are studied. Likewise, different nonlinearities are also considered, with the level of agreements of their solutions are also presented in tabular form to facilitate ease of comparison. Numerical results obtained in all cases revealed that retention of one or two terms in the solution suffices for a reasonable level of accuracy for system with no periodic solution.

1. Introduction

Many dynamical systems, when modeled result into differential equations, mostly nonlinear ordinary differential equations [1, 2]. Dynamical system such as simple pendulum, conservative and non-conservative oscillators, and so on, belong to the dynamical systems with periodic solutions. Many researchers have worked on the solution of this category of problems and proffer reliable solutions to both conservative and non-conservative oscillators. Prominent among the methods of solution are; modified differential transform method for solution of excited nonlinear oscillators under damping effects by [3], nonlinear oscillator with discontinuity was discussed by [4], where He's energy balance method was adopted, [5] used analytical approximation technique to solve strongly nonlinear oscillator problems, just to mention a few.

While on the other hand, there is another category of dynamical systems that do not involve periodic solutions. Such problems work mainly on the principles of moment, where loads are placed on static beams fixed at both ends, is a good example of dynamical system that does not involve periodic solution. A beam is a structural item with a length that is considerably longer than its width and thickness [6]. Solution to such problems were considered in [1,6,7,8]. While perturbation and modified perturbation methods were employed in [6] to solve dynamical problems with no periodic solution and nonlinear oscillators respectively, [3] on the other hand used modified perturbation theory for anharmonic oscillator.

In the present work, the approach proposed in [7] is adopted while using perturbation theory to handle dynamical problems with no periodic solution, but with varying degrees of nonlinearities. It has been shown here too, that after injecting the results of asymptotic expansion in the given problem and resulting system of nonlinear initial value problems solved, retention of few terms in the final series solution suffices for an appreciable level of accuracy.

2. Methodology
2.1. Perturbation Method

Perturbation Method (PM) involves the introduction of a perturbation term into the nonlinear term that occurs in the given differential equation, if it does not come with the problem. The perturbation term, which depends on a small parameter, ε, that is added to the system to bring about corrections [6]. These corrections which are mostly smaller compared to the size of the other quantities, are now calculated using asymptotic series. This method gives best results when applied to non-dynamical system problems with a non-periodic solution (static systems). To illustrate the method, we shall use the nonlinear initial value problem

$$y''(x) + y(x) = y'(x)^2, \quad y(0) = A, \quad y'(0) = 0 \tag{1}$$

To solve (1) by PM, a small nonnegative perturbation term ε is introduced to the nonlinear term $y'(x)^2$. Thus, the IVP in (1) becomes

$$y''(x) + y(x) = \varepsilon y'(x)^2, \quad y(0) = A, \; y'(0) = 0 \tag{2}$$

The asymptotic expansion of $y(x,\varepsilon)$ is given as

$$y(x,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k y_k(x) + O(\varepsilon^{n+1}) \tag{3}$$

The equivalent form of (3) is

$$y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots + \varepsilon^n y_n(x) + O(\varepsilon^{n+1}) \tag{4}$$

Either (3) or (4) is substituted into (2), to get

$$\begin{aligned} y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + O(\varepsilon^3) \\ = \varepsilon\big(y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + O(\varepsilon^3)\big)^2 - y_0(x) - \varepsilon y_1(x) - \varepsilon^2 y_2(x) \\ + O(\varepsilon^3) \end{aligned} \tag{5}$$

$$\begin{aligned} y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + \cdots \\ = \varepsilon(y_0'(x)^2 + 2\varepsilon y_0'(x) y_1'(x) + 2\varepsilon^2 {y_1}'(x)^2 + 2\varepsilon^2 {y_0}'(x) {y_2}'(x) + \cdots) - y_0(x) \\ - \varepsilon y_1(x) - \varepsilon^2 y_2(x) \ldots \end{aligned} \tag{6}$$

$$y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + \cdots$$
$$= -y_0(x) - \varepsilon(y_1(x) - y_0{}'(x)^2) - \varepsilon^2\big(y_2(x) - 2y_0'(x)y_1'(x)\big)$$
$$- \varepsilon^3(y_3(x) - 2y_0'(x)y_2'(x) - y_1{}'(x)^2) + \cdots \quad (7)$$

Comparing the coefficients of various power ε, the IVPs are obtained:

$$\varepsilon^0: \quad y_0''(x) + y_0(x) = 0, \quad y(0) = A, \quad y'(0) = 0 \quad (8)$$
$$\varepsilon^1: \quad y_1''(x) + y_1(x) = y_0{}'(x)^2, \quad y_1(0) = 0, \quad y_1{}'(0) = 0 \quad (9)$$
$$\varepsilon^2: \quad y_2''(x) + y_2(x) = 2y_0'(x)y_1'(x), \quad y_2(0) = 0, \quad y_2'(0) = 0 \quad (10)$$
$$\varepsilon^3: \quad y_3''(x) + y_3(x) = 2y_0'(x)y_2'(x) + y_1{}'(x)^2, \quad y_3(0) = 0, \quad y_3'(0) = 0 \quad (11)$$

What follows is the solution of the system of Equations (8), (9), (10) and (11) using any of *the* known methods. We therefore obtain $y_0(x), y_1(x), y_2(x)$, ... and substitute them in (4) to get $y(x)$ as

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \quad (12)$$

Sometimes, solving for $y_0(x), y_1(x)$ and $y_2(x)$ suffices, and som*e other times,* solving for the first two may suffice. As stated earlier, if the problem being solved is a dynamical system that does not involve periodic solutions then solving for $y_0(x)$ alone or at most solving for $y_0(x)$ and $y_1(x)$ suffices.

3. Numerical Experiments

To illustrate the implementation of Perturbation Method (PM) described in the earlier sections of this paper, we consider a static system that does not have periodic solutions, since this where PM gives optima results, [6].

Problem1 [6]

Consider the structural beam with two hinged ends under a concentrated load at the middle of the span. The governing equation for this system is given by the following IVP:

$$y''(x) - \frac{Fx}{2EI}(1 + y'(x)^2)^{\frac{3}{2}} = 0, \quad (13)$$

together with the initial conditions

$$y(0) = 0, \;\; y'(0) = \frac{FL^2}{16EI}, \quad (14)$$

where $y(x)$ is the beam deflection (vertical displacement), $y'(x)$ is the corresponding slope and x is the time. The constants E and I are the modulus of elasticity and the moment of inertia of the cross section about its axis, respectively. L is the length of the beam and F is the load which is concentrated at the middle of the span.

Solution

In this case the problem did not come with the required small parameter, so it shall be introduced by adding it to the nonlinear term as follows:

$$y''(x) - \frac{Fx}{2EI}(1 + \varepsilon y'(x)^2)^{\frac{3}{2}} = 0, \quad (15)$$

With the conditions: $y(0) = 0$ and $y'(0) = \frac{FL^2}{2EI}$ using the expansion:

$$y(x, \varepsilon) = \sum_{k=0}^{n} \varepsilon^k y_k(x) + O(\varepsilon^{n+1}) \quad (16)$$

In (3), when (3) is written in the form

$$(y''(x))^2 = \frac{F^2x^2}{4E^2I^2}\,(1 + \varepsilon y^2(x)^2)^3, \tag{17}$$

We have

$$\left(\sum_{k=0}^{n} \varepsilon^k\, y_k''(x) + O(\varepsilon^{n+1})\right)^2 = \frac{F^2x^2}{4E^2I^2}\left[1 + \varepsilon\left(\sum_{k=0}^{n} \varepsilon^k y_k''(x) + O(\varepsilon^{n+1})\right)^2\right]^3 \tag{18}$$

$$[y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + O(\varepsilon^3)]^2 = \frac{F^2x^2}{4E^2I^2}[1 + \varepsilon(y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + \cdots)^2]^3 \tag{19}$$

$$\left(y_0''(x)\right)^2 + 2\varepsilon y_0''(x)y_1''(x) + \varepsilon^2\left(y_1''(x))^2 + 2y_0''(x)y_2''(x)\right) + \cdots = \frac{F^2x^2}{4E^2I^2}\Big[1 + 3\varepsilon\left(y_0'(x)\right)^2 + \varepsilon^2\left(3\left(y_0'(x)\right)^4 + 6y_0'(x)y_1'(x)\right) + \varepsilon^3\left(\left(y_0'(x)\right)^6 + 12\left(y_0'(x)\right)^3 y'_1(x) + 3\left(y_1'(x)\right)^2 + 6y_0'(x)y_2'(x)\right) + \cdots\Big] \tag{20}$$

Equating the coefficient of $\varepsilon^0, \varepsilon^1$ and ε^2 in (8) gives:

$$\varepsilon^0: \left(y_0'(x)\right)^2 = \frac{F^2x^2}{4E^2I^2}, \quad y(0) = 0, \qquad y_0'(0) = \frac{FL^2}{16EI} \tag{21}$$

$$\varepsilon^1: 2y_0''(x)y_1''(x) = \frac{3F^2x^2}{4E^2I^2}\left(y_0'(x)\right)^2, \quad y_1(0) = 0, \;\; y'_0 = 0 \tag{22}$$

$$\varepsilon^2: \left(y_1''(x)\right)^2 + 2y_0''(x)y_2''(x) = \frac{3F^2x^2}{4E^2I^2}\left(3\left(y_0'(x)\right)^4 + 2y_0'(x)y_1'(x)\right), y_2(0) = y'_2(0) = 0 \tag{23}$$

Solving (21) and (22) shall suffice, as the solution to the (23) will be cumbersome and may therefore not be necessary.
From (21),

$$y_0''(x) = \frac{Fx}{2EI}\;, \qquad y_0(0) = 0, \qquad y'_0(0) = \frac{FL^2}{16EI} \tag{24}$$

$$y_0'(x) = \frac{Fx^2}{4EI} + c_1, \tag{25}$$

Using the associated condition, we have

$c_1 = \frac{FL^2}{16EI}$.

Substituting for c_1 in (25), we get

$$y_0'(x) = \frac{Fx^2}{4EI} + \frac{FL^2}{16EI} \tag{26}$$

$$y_0(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI} + c_2 \tag{27}$$

Using the condition $y_0 = 0$, we have

$c_2 = 0$.

Thus, (27) becomes

$$y_0(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI} \tag{28}$$

(28) is taken as the solution of (21), not losing focus on the fact that (21) will have two answers that only differs in sign. Since (28) satisfies the subsidiary conditions in (21), it is accepted as the solution.

We now proceed with the solution of (10) as follows:

$$y_1''(x) = \frac{3F^2x^2}{4E^2I^2}\left(y'_0(22)\right)^2, \qquad y_1(0) = y'_1(0) = 0$$

$$2\left(\frac{Fx}{2EI}\right)y_1''(x) = \frac{3F^2x^2}{4E^2I^2}\left(\frac{Fx^2}{4EI} + \frac{FL^2}{16EI}\right)$$

$$y_1''(x) = \frac{3F^2x^2}{4E^2I^2} \cdot \frac{EI}{Fx}\left(\frac{Fx^2}{4EI} + \frac{FL^2}{16EI}\right)$$

$$y_1''(x) = \frac{3Fx}{4EI}\left(\frac{Fx^2}{4EI} + \frac{FL^2}{16EI}\right)$$

$$y_1'(x) = \frac{3F^2x^4}{64E^2I^2} + \frac{3F^2L^2x^2}{128E^2I^2} + c_3 \tag{29}$$

Using the associated conditions, (22) gives

$c_3 = 0.$

Thus,

$$y_1'(x) = \frac{3F^2x^4}{64E^2I^2} + \frac{3F^2L^2x^2}{128E^2I^2}$$

And

$$y_1(x) = \frac{3F^2x^5}{320E^2I^2} + \frac{F^2L^2x^3}{128E^2I^2} + c_4 \tag{30}$$

Using $y_1(0) = 0$, (30) gives

$c_4 = 0.$

Thus,

$$y_1(x) = \frac{3F^2x^5}{320E^2I^2} + \frac{3F^2L^2x^3}{128E^2I^2} \tag{31}$$

Hence, the solution of (15) is given by

$$y(x) = y_0(x) + \varepsilon y_1(x) + \cdots \tag{32}$$

As

$$y(x) = \frac{FL^2x}{16EI} + \frac{Fx^3}{12EI} + \varepsilon\left(\frac{3F^2L^2x^3}{128E^2I^2} + \frac{3F^2x^5}{320E^2I^2}\right) + \cdots \tag{33}$$

Since ε is a very small nonnegative constant, taking $\varepsilon = 1$ in this case yields

$$y(x) = \frac{FL^2x}{16EI} + \frac{Fx^3}{12EI} + \frac{3F^2L^2x^3}{128E^2I^2} + \frac{3F^2x^5}{320E^2I^2}.$$

Problem 2 [6]

Solve the BVP in Problem 1 by setting

$$(1 + y'(x)^2)^{\frac{3}{2}} \approx 1 + \frac{3}{2}y'^{(x)^2}. \quad (34)$$

And compare your results with these obtained in Problem 1.

Solution

The new problem is

$$y''(x) - \frac{Fx}{2EI}\left(1 + \frac{3}{2}y'(x)^2\right) = 0, \quad (35)$$

with the conditions

$$y(0) = 0 \quad and \quad y'(0) = \frac{FL^2}{16EI} \quad (36)$$

To solve this problem by permutation method, the small positive parameter ε is introduced to the nonlinear term in (35). Thus, we have

$$y''(x) - \frac{Fx}{2EI}\left(1 + \frac{3}{2}\varepsilon y'(x)^2\right) = 0 \quad (37)$$

Using

$$y(x,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k y_k(x) + O(\varepsilon^{n+1}) \quad (38)$$

in (35), we have

$$\sum_{k=0}^{n} \varepsilon^k y_k''(x) + O(\varepsilon^{n+1}) = \frac{Fx}{2EI}\left[1 + \frac{3}{2}\varepsilon\left(\sum_{k=0}^{n} \varepsilon^k y_k'(x) + O(\varepsilon^{n+1})\right)^2\right] \quad (39)$$

$$y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + O(\varepsilon)^2 = \frac{Fx}{2EI}\left[1 + \frac{3}{2}\varepsilon(y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + O(\varepsilon)^3)^2\right] \quad (40)$$

$$y_0''(x) + \varepsilon y_1''(x) + \varepsilon^2 y_2''(x) + \cdots = \frac{Fx}{2EI}\left(1 + \frac{3\varepsilon\left(y_0'(x)\right)^2}{2} + 3\varepsilon^2 y_0'(x)y_1'(x) + \cdots\right) \quad (41)$$

Taking the coefficients of various powers of ε, we have

$$\varepsilon^0: \quad y''_0(x) = \frac{Fx}{2EI}, \quad y_0(0) = 0, \quad y_0'(0) = \frac{FL^2}{16EI} \quad (42)$$

$$\varepsilon^1: \quad y''_1(x) = \frac{3Fx}{4EI} y_0'(x)^2, \;\; y_1(0) = 0, \quad y_1'(0) = 0 \tag{43}$$

$$\varepsilon^2: \;\; y''_2(x) = \frac{3Fx}{2EI} y_0'(x) y_1'(x), \quad y_2(0) = 0, \qquad y_2'(0) = 0 \tag{44}$$

Solving (42), (43) and (44) successively, we have

$$y_0''(x) = \frac{Fx}{2EI} \tag{45}$$

$$y''_0(x) = \frac{Fx^2}{4EI} + c_1 \tag{46}$$

Using the associated conditions, we get

$$c_1 = \frac{FL^2}{16EI} \tag{47}$$

Hence (46) becomes

$$y_0'(x) = \frac{Fx^2}{4EI} + \frac{FL^2}{16EI} \tag{48}$$

$$y_o(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI} + c_2 \tag{49}$$

Using $y_0(0) = 0$, we have

$$c_2 = 0,$$

hence (49) becomes

$$y_0(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI}. \tag{50}$$

For the solution of (43), we proceed as follows:

$$y''_1(x) = \frac{3Fx}{4EI}\left(\frac{Fx^2}{4EI} + \frac{FL^2}{16EI}\right)^2,$$

$$y''_1(x) = \frac{3F^2x^3}{16E^2I^2} + \frac{3F^3x^3L^2}{128E^3I^3} + \frac{3F^3L^4x}{64E^3I^3} \tag{51}$$

$$y_1'(x) = \frac{3F^2x^4}{64E^2I^2} + \frac{3F^3x^4L^2}{512E^3I^3} + \frac{3F^3L^3x^2}{128E^3I^3} + c_3 \tag{52}$$

Using the attached conditions, we have

$$c_3 = 0.$$

Thus (52) reduces to

$$y_1'(x) = \frac{3F^2x^4}{64E^2I^2} + \frac{3F^3x^4L^2}{512E^3I^3} + \frac{3F^3L^3x^2}{128E^3I^3} \quad (53)$$

$$y_1(x) = \frac{3F^2x^5}{320E^2I^2} + \frac{3F^3x^5L^2}{3060E^3I^3} + \frac{F^3L^3x^3}{128E^3I^3} + c_4 \quad (54)$$

Using the associated condition here again gives

$$c_4 = 0.$$

Hence,

$$y_1(x) = \frac{3F^2x^5}{320E^2I^2} + \frac{3F^3x^5L^2}{3060E^3I^3} + \frac{F^3L^3x^3}{128E^3I^3} \quad (43)$$

Solving (44) becomes cumbersome, hence the first two results shall suffice here again. Therefore,

$$y(x) = y_0(x) + \varepsilon y_1(x) + \cdots \quad (44)$$

$$y(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI} + \varepsilon\left(\frac{3F^2x^5}{320E^2I^2} + \frac{3F^3x^5L^2}{3060E^3I^3} + \frac{F^3L^3x^3}{128E^3I^3}\right) + \cdots \quad (45)$$

Using $\varepsilon = 1$, we have

$$y(x) = \frac{Fx^3}{12EI} + \frac{FL^2x}{16EI} + \frac{3F^2x^5}{320E^2I^2} + \frac{3F^3x^5L^2}{3060E^3I^3} + \frac{F^3L^3x^3}{128E^3I^3}. \quad (38)$$

4. Results and Discussion

Table 1: Results for $y(x) = y_0(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1	1	500	100	0.029166710	0.029166710
2	1	500	100	0.15833310	0.158333100
3	1	500	100	0.48751000	0.48751000
4	1	500	100	1.11667100	1.11667100
5	1	500	100	2.145583100	2.14583100

Table 2: Results for $y(x) = y_0(x) + y_1(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1	1	500	100	0.029854210	0.02961210
2	1	500	100	0.172833100	0.17108410
3	1	500	100	0.587063100	0.58221810
4	1	500	100	1.52067100	1.51271000
5	1	500	100	3.35677100	3.35003100

Table 3: Results for $y(x) = y_0(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1.0	3	500	100	0.12916710	0.12916710
2.0	3	500	100	0.35833310	0.35833310

3.0	3	500	100	0.7875100	0.78751000
4.0	3	500	100	1.51667100	1.51667100
5.0	3	500	100	2.64583100	2.64583100

Table 4: Results for $y(x) = y_0(x) + y_1(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1	3	500	100	0.13235410	0.1313100
2	3	500	100	0.39283310	0.38609210
3	3	500	100	0.95456310	0.94134100
4	3	500	100	2.08067100	2.08095100
5s	3	500	100	4.16927100	4.24923100

Table 5: Results for $y(x) = y_0(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1	5	500	100	0.32916710	0.32916710
2	5	500	100	0.75833310	0.75833310
3	5	500	100	1.38751000	1.38751000
4	5	500	100	2.31667100	2.316671000
5	5	500	100	3.6458310	3.64583100

Table 6: Results for $y(x) = y_0(x) + y_1(x)$ for Problems 1 and 2

x	L	EI	F	Problem 1	Problem 2
1	5	500	100	0.33735410	0.33755100
2	5	500	100	0.83283310	0.83910810
3	5	500	100	1.68956100	1.7372110
4	5	500	100	3.20067100	3.40145100
5	5	500	100	5.79427100	5.40702100

4.1. Discussion

The two problems considered are different only in the nonlinearity parts as indicated in Problem 2. The problems were solved by perturbation method with the usual introduction of the perturbation parameter, ε, to nonlinear terms. The solutions are presented by retaining the first term if the series only for the two problems, and also by retaining the first two terms of the series. Solving for the third term in the two cases presented very cumbersome situations and were therefore abandoned. Tables 1 and 2, 3 and 4, and 5 and 6 presents the numerical results obtained using the same set of values assigned to the terms and L taken three values respectively. All the values assigned to the constants are adopted from [6]. It is easily seen in the tables that the solutions with only $y_0(x)$ retained produced the same results for the two problems, as we have them in Tables 1, 3, and 5. On the other hand, with $y_0(x)$ and $y_1(x)$ retained the results are still close, but not completely the same. As stated earlier, retaining one or two terms in the series suffices, since there are no significant differences in the results with one term and two terms retained.

5. Conclusion

Perturbation Method (PM) has been shown to have performed very well for the problems considered in this work. The results obtained for the two nonlinearities are very close and satisfactory results are realized by retaining one or two terms in the solution series. It is therefore sufficient to affirm that PM is very suitable for solutions of dynamical systems that do not possess periodic solution.

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