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Analysis of Thick Anisotropic Plate Through Exact Approach Using Third Order Shear Deformation Theory

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ABSTRACT

This work presents the analysis of thick anisotropic plate through exact approach using Third Order Shear Deformation Theory. Total potential energy was formed based on the refined plate theory assumptions. Displacement field, kinematic relations, constitutive relations and stress displacement relations were derived from the deformed section of a thick rectangular anisotropic plate. Strain energy was formed by substituting the kinematic relations and stress-displacement relations into the universal strain energy equation. By the addition of the external work to the strain energy equation, total potential energy functional for the analysis of thick anisotropic rectangular plate was obtained. The total potential energy functional was minimized by differentiating it with respect to the changes in out-plane deflection, δw, shear deformation rotation in x direction, $\delta\phi_x$, and shear deformation rotation in y direction, $\delta\phi_y$. This yielded the governing equation and two compatibility equations of thick anisotropic rectangular plate. A third order polynomial shear deformation was employed in the governing and compatibility equations to obtain the displacement functions (deflection, w, shear deformation rotation in x direction, ϕ_x, and shear deformation rotation in y direction, ϕ_y). These displacement functions (w, ϕ_x, ϕ_y) obtained satisfied the specified boundary conditions and it gave the unique displacement functions for each of the four plate boundary conditions SSSS, SCFS, CCFS and SCFC solved. The stiffness coefficients ($K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8$) were calculated for each of the four plate boundary conditions. The formulas for calculating the coefficients of the displacements were combined with elastic equations to determine the formulas which were used in calculating for displacements (u, v and w) and non-dimensional stresses ($\sigma_{RR}, \sigma_{QQ}, \tau_{RQ}, \tau_{RS}$ and τ_{QS}) at "0^0" angle fiber orientation and various span to thickness ratio, α(5, 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100) and for all the four boundary conditions. These formulas were used to analyze some typical anisotropic rectangular thick plates by the help of a functional excel worksheet program. The numerical results obtained for displacement (w) and stresses ($\overline{\sigma_{xx}}$ and $\overline{\sigma_{yy}}$) at aspect ratio of 1.0 and span to thickness ratio of, 20.0, 10.0, and 7.14286, in this study, when compared with the results of Shimpi and Patel showed percentage difference of 0.59, 1.47, 2.70; 0.62, 1.20, 1.91 and 1.31, 0.97, 3.91% which is in good agreement. Hence the developed method is recommended for analyzing thick rectangular anisotropic plates.

1. Introduction

Technological progress is associated with continuous improvement of existing material properties and this has led to the expansion of structural material classes and types. Usually new materials emerge due to the need to improve structural efficiency and performance. These new materials in turn provide opportunities to develop outdated structures and technologies, and also create new problems and tasks to engineers and material scientists. One of the best manifestations of these related processes is the development of the composite structural elements which are associated with the anisotropic structural plate, to which this study is devoted.

Composite materials emerged in the middle of the twentieth century as a promising class of engineering materials providing new prospects for modern technology. Broadly speaking, any material consisting of two or more components with different properties and distinct boundaries between the components can be referred to as a composite [1].

The sudden increase in the use of anisotropic or composite materials in many types of engineering structures (e.g., high rise structures, aerospace, underwater structures, automotive, electronic circuit board, medical prosthetic devices and sports equipment) and the number of journals and research papers published in the last two decades attest to the fact that there has been a major effort to develop composite material systems, and to analyze and design structural components made from composite materials [2]. The production of anisotropic material involves chemists, electrical engineers, chemical engineers, material scientists, mechanical engineers, and structural engineers. Structural engineers deal mainly with the analysis and design of these anisotropic materials [3].

Anisotropic plates are plates with different resistance to mechanical actions in different directions. This implies that anisotropic plates are directionally dependent as opposed to isotropic plates that implies identical properties in all directions. Examples of anisotropic plates are aviation plywood, delta wood, coated aluminum plate, alloyed metal plates and a number of other materials [4, 5].

Works on refined plate theory have been characterized by the use of trigonometric displacement function. Many scholars have obtained the closed form solutions and others have obtained approximate solution using assumed displacement functions in energy method. However, one thing that is common in them all is the use of trigonometric displacement functions to approximate the deformed shapes of the plates. Others have applied the assumed polynomial displacement functions in numerical methods like finite element method and differential quadrature element methods [6, 7, & 8]. The major flaw in their traditional refined plate theory (i.e. Third order or higher order shear deformation theory) is the assumption of their displacement functions in thick anisotropic plate analysis. It is believed that these assumptions have not been solved to ascertain their validity or correctness in thick anisotropic plate analysis [9].

Because of the complexity involved in handling thick anisotropic plates, engineers usually resort to thin isotropic plate or even thick isotropic plate despite the numerous shortcomings. Isotropic plate assumes that the material properties at a point are the same in all directions. However, certain materials display direction-dependent properties; consequently, these materials are referred to as anisotropic materials. When an anisotropic material is stressed in one of the principal directions, the lateral deformations in the other principal directions could be smaller or larger than the deformation in the direction of the applied stress depending on the material properties. Idealization of a thick anisotropic plate as a thin isotropic plate always underestimates the stresses in the plate. The consequence of using these erroneous stresses in design and construction is structural failure and sometimes total collapse. Also earlier works on thick anisotropic plates are based mainly on trigonometric and assumed displacement functions. It is rare to see work on anisotropic thick plate analysis that determined the exact polynomial shape function from the integration of governing equation of equilibrium and compatibility equations of thick anisotropic plate [3 &10]. Thus, it can be said that earlier works on the bending analysis of thick anisotropic plates have yielded approximate results, since it cannot be said that the displacement functions used are exact [10]. The need to approach anisotropic thick plate analysis from the perspective of determining the exact displacement functions through integration of the governing equation prompted the present study. This inability to arrive at the exact displacement function has been identified as a gap in literature that has to be filled up. To cover this gap in anisotropic thick plate analysis is the primary motivation of the present study.

2. Methodology

2.1. Formulation of total potential energy functional

The total potential energy functional for a thick anisotropic rectangular plate has been formulated as shown. In formulating the total potential energy the work was based on the refined plate theory assumptions.

a. Determination of Displacement field

In-plane displacements, u and v of refined plate theory are defined as shown:

$$u = u_c + u_s \qquad 1$$

$$v = v_c + v_s \qquad 2$$

Aspect ratio, (β = b/a), (R = x/a; Q = y/b; S = z/t) are the non-dimensional forms of the orthogonal axes

The in-plane displacements, u_c, v_c and u_s, v_s, where subscripts “c” and “s” stands for classical and transverse are as presented in Equations 3, 4, 5 and 6.

$$u_c = -z\theta_{cx} = -z\frac{dw}{dx} = -\frac{St}{a}\frac{dw}{dR} \qquad 3$$

$$v_c = -z\theta_{cy} = -z\frac{dw}{dy} = -\frac{St}{b}\frac{dw}{dR} = -\frac{St}{\beta a}\frac{dw}{dQ} \qquad 4$$

$$u_s = F(z)\theta_{sx} \qquad 5$$

$$v_s = F(z)\theta_{sy} \qquad 6$$

Also, F(z) stands for the third order shear deformation model presented as:

$$F(z) = z - \frac{4}{3}.\frac{z^3}{t^2} = z\left(1 - \frac{4}{3}\left[\frac{z}{t}\right]^2\right) \qquad 7a$$

The model in a non-dimensional form is as defined:

$$F = F(s) = t\left(S - \frac{4}{3}S^3\right) \qquad 7b$$

Where:

$$F = tH \qquad 7c$$

$$H = S - \frac{4}{3}S^3 \qquad 7d$$

Equations 3 and 5 can be combined to obtain Equation 8a

$$u = -\frac{St}{a}\frac{dw}{dR} + F(z).\emptyset_x \qquad 8a$$

Also, Equations 4 and 6 can be combined to obtain Equation 8b

$$v = -\frac{St}{\beta a}\frac{dw}{dQ} + F(z).\emptyset_y \qquad 8b$$

Equation 7c can be substituted into Equations 8a and 8b to obtain Equations 8c and 8d

$$u = \frac{t}{a}\left[-S\frac{\partial w}{\partial R} + Ha.\emptyset_x\right] \qquad 8c$$

$$v = \frac{t}{a\beta}\left[-S\frac{\partial w}{\partial Q} + \beta Ha.\emptyset_y\right] \qquad 8d$$

b. Determination of kinematic relations

Equations 9, 10, 11, 12 and 13 as presented are Equations of strain – displacement relations

$$\varepsilon_R = \frac{\partial u}{\partial x} = \frac{\partial u}{a\partial R} = \frac{t}{a^2}\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R}\right] \qquad 9$$

$$\varepsilon_Q = \frac{\partial v}{\partial y} = \frac{\partial v}{a\beta\partial Q} = \frac{t}{\beta^2 a^2}\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q}\right] \qquad 10$$

$$\gamma_{RQ} = \varepsilon_{RQ} + \varepsilon_{QR} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= \frac{t}{\beta a^2}\left[-S\frac{\partial^2 w}{\partial R\partial Q} + Ha.\frac{\partial \phi_x}{\partial Q}\right] + \frac{t}{\beta a^2}\left[-S\frac{\partial^2 w}{\partial R\partial Q} + H\beta a.\frac{\partial \phi_y}{\partial R}\right]. \text{That is:}$$

$$\gamma_{RQ} = \frac{t}{\beta a^2}\left[-2S\frac{\partial^2 w}{\partial R\partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right] \qquad 11$$

$$\gamma_{RS} = \varepsilon_{RS} + \varepsilon_{SR} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{a}\left[-\frac{\partial w}{\partial R} + a\frac{\partial H}{\partial S}.\emptyset_x\right] + \frac{1}{a}\frac{\partial w}{\partial R}. \text{That is:}$$

$$\gamma_{RS} = \frac{\partial H}{\partial S}.\emptyset_x \qquad 12$$

$\gamma_{QS} = \frac{\partial H}{\partial S}.\emptyset_y = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{\beta a}\left[-\frac{\partial w}{\partial Q} + \beta a\frac{\partial H}{\partial S}.\emptyset_y\right] + \frac{1}{\beta a}.\frac{\partial w}{\partial Q}$. That is:

$$\gamma_{QS} = \frac{\partial H}{\partial S}.\emptyset_y \tag{13}$$

c. Determination of constitutive relations

$$\begin{bmatrix}\sigma_R\\ \sigma_Q\\ \tau_{RQ}\\ \tau_{RS}\\ \tau_{QS}\end{bmatrix} = \frac{E_0}{1-\mu_{12}\mu_{21}}\begin{bmatrix}B_{11} & B_{12} & B_{13} & 0 & 0\\ B_{21} & B_{22} & B_{23} & 0 & 0\\ B_{31} & B_{32} & B_{33} & 0 & 0\\ 0 & 0 & 0 & B_{44} & 0\\ 0 & 0 & 0 & 0 & B_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_R\\ \varepsilon_Q\\ \gamma_{RQ}\\ \gamma_{RS}\\ \gamma_{QS}\end{bmatrix} \tag{14}$$

Where:

E_0 is the reference elastic modulus. It can be E_1 or E_2; $m = Cos\,\theta$; $n = Sin\,\theta$

$$B_{11} = m^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + n^4 d_{22} \tag{15}$$

$$B_{12} = d_{12}(n^4 + m^4) + m^2n^2(d_{11} + d_{22} - 4d_{33}) \tag{16}$$

$$B_{13} = m^3n(d_{11} - d_{12} - 2d_{33}) + mn^3(d_{12} - d_{22} + 2d_{33}) \tag{17}$$

$$B_{22} = n^4 d_{11} + 2m^2n^2(d_{12} + 2d_{33}) + m^4 d_{22} \tag{18}$$

$$B_{23} = mn^3 d_{11} - m^3n d_{22} + (m^3n - mn^3)(d_{12} + 2d_{33}) \tag{19}$$

$$B_{33} = m^2n^2(d_{11} - 2d_{12} + d_{22} - 2d_{33}) + d_{33}(m^4 + n^4) \tag{20}$$

$$B_{44} = d_{44};\ B_{55} = d_{55}; B_{21} = B_{12}; B_{31} = B_{13};\ B_{32} = B_{23} \tag{21}$$

$$d_{11} = E_1/E_0 \tag{22}$$

$$d_{12} = E_2\,.\,\mu_{12}/E_0 \tag{23}$$

$$d_{21} = E_1\,.\,\mu_{21}/E_0 \tag{24}$$

$$d_{22} = E_{22}/E_0 \tag{25}$$

$$d_{33} = G_{12}(1 - \mu_{12}\mu_{21})/E_0 \tag{26}$$

$$d_{44} = G_{13}(1 - \mu_{12}\mu_{21})/E_0 \tag{27}$$

$$d_{55} = G_{23}(1 - \mu_{12}\mu_{21})/E_0 \tag{28}$$

Substituting Equations 9 to 13 into Equation 14 gives each stress component as:

$$\sigma_R = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}.\left(B_{11}.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R}\right] + \frac{B_{12}}{\beta^2}.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q}\right] + \frac{B_{13}}{\beta}.\left[-2S\frac{\partial^2 w}{\partial R \partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right]\right) \quad 29$$

$$\sigma_Q = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}.\left(B_{21}.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R}\right] + \frac{B_{22}}{\beta^2}.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q}\right] + \frac{B_{23}}{\beta}.\left[-2S\frac{\partial^2 w}{\partial R \partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right]\right) \quad 30$$

$$\tau_{RQ} = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}.\left(B_{31}.\left[-S\frac{\partial^2 w}{\partial R^2} + Ha.\frac{\partial \phi_x}{\partial R}\right] + \frac{B_{32}}{\beta^2}.\left[-S\frac{\partial^2 w}{\partial Q^2} + Ha\beta.\frac{\partial \phi_y}{\partial Q}\right] + \frac{B_{33}}{\beta}.\left[-2S\frac{\partial^2 w}{\partial R \partial Q} + Ha.\left(\frac{\partial \phi_x}{\partial Q} + \beta.\frac{\partial \phi_y}{\partial R}\right)\right]\right) \quad 31$$

$$\tau_{RS} = \frac{E_0}{1-\mu_{12}\mu_{21}}.B_{44}.\left[\frac{\partial H}{\partial S}\right].\emptyset_x = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}.B_{44}.\left[\frac{a^2}{t}.\frac{\partial H}{\partial S}\right].\emptyset_x \quad 32$$

$$\tau_{QS} = \frac{E_0}{1-\mu_{12}\mu_{21}}.B_{55}.\left[\frac{\partial H}{\partial S}\right].\emptyset_y = \frac{E_0 t}{[1-\mu_{12}\mu_{21}]a^2}.B_{55}.\left[\frac{a^2}{t}.\frac{\partial H}{\partial S}\right].\emptyset_y \quad 33$$

d. The total potential energy functional

The total potential energy functional is given as:

$$\Pi = \frac{abt}{2}\int_0^1\int_0^1\int_{-0.5}^{0.5}\left(\sigma_R\varepsilon_R + \sigma_R\varepsilon_R + \tau_{RQ}\gamma_{RQ} + \tau_{RS}\gamma_{RS} + \tau_{QS}\gamma_{QS}\right) dR\, dQ\, dS - qab\int_0^1\int_0^1 w\, dR\, dQ \quad 34$$

Substituting Equations 9 to 13 and Equations 29 to 33 into Equations 34 gives:

$$\Pi = \frac{\mathrm{abD_0}}{2a^4}.\int_0^1\int_0^1\Bigg\{\Bigg\{\mathrm{B_{11}}.\left[\left(\frac{\partial^2 w}{\partial R^2}\right)^2 - 2g_2a.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R} + g_3a^2.\left(\frac{\partial\phi_x}{\partial R}\right)^2\right]$$
$$+\frac{\mathrm{B_{12}}}{\beta^2}.\left[2\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2 - g_2\frac{a}{\beta}\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q} - g_2a\beta^2\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_x}{\partial R} - g_2a.\frac{\partial\phi_x}{\partial R}.\frac{\partial^2 w}{\partial Q^2}\right.$$
$$\left. - g_2a\beta.\frac{\partial^2 w}{\partial R^2}.\frac{\partial\phi_y}{\partial Q} + 2g_3a^2\beta.\frac{\partial\phi_x}{\partial R}.\frac{\partial\phi_y}{\partial Q}\right]$$
$$+\frac{B_{13}}{\beta}.\left[4\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial^2 w}{\partial R^2} - 2g_2a.\left(\frac{\partial\phi_x}{\partial Q} + \beta.\frac{\partial\phi_y}{\partial R}\right).\frac{\partial^2 w}{\partial R^2} - 4g_2a\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial R}\right.$$
$$\left. + 2g_3a^2.\left(\frac{\partial\phi_x}{\partial Q} + \beta.\frac{\partial\phi_y}{\partial R}\right).\frac{\partial\phi_x}{\partial R}\right]$$
$$+\frac{\mathrm{B_{22}}}{\beta^4}.\left[\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 - 2g_2a\beta.\frac{\partial^2 w}{\partial Q^2}.\frac{\partial\phi_y}{\partial Q} + g_3a^2\beta^2.\left(\frac{\partial\phi_y}{\partial Q}\right)^2\right]$$
$$+\frac{B_{23}}{\beta^3}.\left[4\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial^2 w}{\partial Q^2} - 2g_2a.\left(\frac{\partial\phi_x}{\partial Q} + \beta.\frac{\partial\phi_y}{\partial R}\right).\frac{\partial^2 w}{\partial Q^2} - 4g_2a\beta.\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial Q}\right.$$
$$\left. + 2g_3a^2\beta.\left(\frac{\partial\phi_x}{\partial Q} + \beta.\frac{\partial\phi_y}{\partial R}\right).\frac{\partial\phi_y}{\partial Q}\right]$$
$$+\frac{\mathrm{B_{33}}}{\beta^2}.\left[4\left(\frac{\partial^2 w}{\partial R\partial Q}\right)^2 - 2g_2a.\left(\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_x}{\partial Q} + \beta\frac{\partial^2 w}{\partial R\partial Q}.\frac{\partial\phi_y}{\partial R}\right)\right.$$
$$\left. + g_3a^2.\left(\left(\frac{\partial\phi_x}{\partial Q}\right)^2 + 2\beta\frac{\partial\phi_x}{\partial Q}.\frac{\partial\phi_y}{\partial R} + \beta^2.\left(\frac{\partial\phi_y}{\partial R}\right)^2\right)\right] + a^2\mathrm{B_{44}}.\left(\frac{a}{t}\right)^2.g_4.\varnothing_x{}^2$$
$$+ a^2\mathrm{B_{55}}.\left(\frac{a}{t}\right)^2.g_4.\varnothing_y{}^2\Bigg\} - 2\frac{\mathrm{q}a^4}{\mathrm{D_0}}w\Bigg\}\,\mathrm{dR\,dQ} \qquad 35$$

Where:

$$\mathrm{D_0} = \frac{\mathrm{E_0t^3}}{12[1-\mu_{12}\mu_{21}]} \qquad 36$$

e. The governing equation and compatibility equations

Differentiating Equation 35 with respect to w, θ_x and θ_y gives the governing equation and compatibility equations respectively.

$$\frac{\mathrm{d}\Pi}{\mathrm{dw}} = \frac{\mathrm{d}\Pi}{\mathrm{d}\varnothing_x} = \frac{\mathrm{d}\Pi}{\mathrm{d}\varnothing_y} = 0 \qquad 37$$

That is:

$$\frac{d\prod}{dw} = \int_0^1\int_0^1 \left\{ B_{11}.\frac{\partial^4 w}{\partial R^4} + \frac{2}{\beta^2}.B_{xy}\frac{\partial^4 w}{\partial R^2\ \partial Q^2} + \frac{B_{22}}{\beta^4}.\frac{\partial^4 w}{\partial Q^4} + 4\frac{B_{13}}{\beta}.\frac{\partial^4 w}{\partial R^3\,\partial Q} + 4\frac{B_{23}}{\beta^3}.\frac{\partial^4 w}{\partial R\,\partial Q^3} \right.$$
$$- \frac{g_2 a}{2}[2B_{11} + B_{12}]\frac{\partial^3\phi_x}{\partial R^3} - \frac{g_2 a}{2\beta^2}.B_{xy}\frac{\partial^3\phi_x}{\partial R\ \partial Q^2} - 3g_2 a.\frac{B_{13}}{\beta}\frac{\partial^3\phi_x}{\partial R^2\,\partial Q}$$
$$- \frac{g_2 a}{2\beta^3}[B_{12} + 2B_{22}]\frac{\partial^3\phi_y}{\partial Q^3} - \frac{g_2 a}{2\beta}B_{xy}\frac{\partial^3\phi_y}{\partial R^2\,\partial Q} - 3g_2 a.\frac{B_{23}}{\beta^2}\frac{\partial^3\phi_y}{\partial R\,\partial Q^2} - g_2 a.B_{13}.\frac{\partial^3\phi_y}{\partial R^3}$$
$$\left. - \frac{g_2 a}{\beta^3}.B_{23}.\frac{\partial^3\phi_x}{\partial Q^3} - \frac{qa^4}{D_0}\right\} dR\, dQ = 0 \qquad 38$$

$$\frac{d\prod}{d\emptyset_x} = B_{11}.\left[-g_2 a.\frac{\partial^3 w}{\partial R^3} + g_3 a^2.\frac{\partial^2\phi_x}{\partial R^2}\right] + \frac{B_{12}}{2\beta^2}.\left[-g_2 a\beta^2\frac{\partial^3 w}{\partial R^3} - g_2 a.\frac{\partial^3 w}{\partial R\ \partial Q^2} + 2g_3 a^2\beta.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]$$
$$+ \frac{B_{13}}{\beta}.\left[-g_2 a.\frac{\partial^3 w}{\partial Q\ \partial R^2} - 2g_2 a\frac{\partial^3 w}{\partial Q\ \partial R^2} + 2g_3 a^2.\frac{\partial^2\phi_x}{\partial R\partial Q} + g_3 a^2.\beta.\frac{\partial^2\phi_y}{\partial R^2}\right]$$
$$+ \frac{B_{23}}{\beta^3}.\left[-g_2 a.\frac{\partial^3 w}{\partial Q^3} + g_3 a^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$
$$+ \frac{B_{33}}{\beta^2}.\left[-g_2 a.\frac{\partial^3 w}{\partial R\ \partial Q^2} + g_3 a^2.\frac{\partial^2\phi_x}{\partial Q^2} + g_3 a^2.\beta\frac{\partial^2\phi_y}{\partial R\partial Q}\right] + a^2 B_{44}.\left(\frac{a}{t}\right)^2.g_4.\emptyset_x$$
$$= 0 \qquad 39$$

$$\frac{d\prod}{d\emptyset_y} = \frac{B_{12}}{2\beta^2}.\left[-g_2\frac{a}{\beta}\frac{\partial^3 w}{\partial Q^3} - g_2 a\beta.\frac{\partial^3 w}{\partial R^2\partial Q} + 2g_3 a^2\beta.\frac{\partial^2\phi_x}{\partial R\partial Q}\right]$$
$$+ \frac{B_{13}}{\beta}.\left[-g_2 a.\beta.\frac{\partial^3 w}{\partial R^3} + g_3 a^2.\beta.\frac{\partial^2\phi_x}{\partial R^2}\right] + \frac{B_{22}}{\beta^4}.\left[-g_2 a\beta.\frac{\partial^3 w}{\partial Q^3} + g_3 a^2\beta^2.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$
$$+ \frac{B_{23}}{\beta^3}.\left[-g_2 a.\beta.\frac{\partial^3 w}{\partial R\ \partial Q^2} - 2g_2 a\beta.\frac{\partial^3 w}{\partial R\partial Q^2} + g_3 a^2\beta.\frac{\partial^2\phi_x}{\partial Q^2} + 2g_3 a^2\beta^2.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]$$
$$+ \frac{B_{33}}{\beta^2}.\left[-g_2 a.\beta\frac{\partial^3 w}{\partial R^2\partial Q} + g_3 a^2.\beta\frac{\partial^2\phi_x}{\partial R\partial Q} + g_3 a^2.\beta^2.\frac{\partial^2\phi_y}{\partial R^2}\right] + a^2 B_{55}.\left(\frac{a}{t}\right)^2.g_4.\emptyset_y$$
$$= 0 \qquad 40$$

Equations 38, 39 and 40 are the governing equation of equilibrium of forces, compatibility equation of displacements in x-z plane and compatibility equation of displacements in y-z plane respectively.

f. Solutions of governing equation and compatibility equations

Solving Equations 38, 39 and 40 gives:

$$w = A_1 h \qquad 41a$$

$$w = (\alpha_0 + \alpha_1 R + \alpha_2 R^2 + \alpha_3 R^3 + \alpha_4 R^4)(\lambda_0 + \lambda_1 Q + \lambda_2 Q^2 + \lambda_3 Q^3 + \lambda_4 Q^4) \qquad 41b$$

$$\phi_x = \frac{A_2}{a}.\frac{\partial h}{\partial R} \qquad 42$$

$$\emptyset_y = \frac{A_3}{a\beta}.\frac{\partial h}{\partial Q} \qquad 43$$

Satisfying the boundary condition for SSSS plates gives their distinct deflection equation respectively as:

$$w = A_1 h = \alpha_3\alpha_4(R - 2R^3 + R^4)(Q - 2Q^3 + \lambda_4 Q^4) \qquad \{for\ SSSS\} \qquad 41c$$

Substituting Equations 41a, 42 and 43 into Equation 35 gives:

$$\begin{aligned}\Pi = \frac{ab}{2a^4}.\Bigg\{\Bigg\{ & B_{11}.\left[A_1{}^2 - 2g_2A_1A_2. + g_3A_2{}^2\right]k_1 + \frac{(B_{12} + 2B_{33})}{\beta^2}.\left[2A_1{}^2 - g_2A_1A_2 - g_2A_1A_3\right].k_2 \\ & + 2\frac{[B_{12} + B_{33}]}{\beta^2} g_3\, A_2A_3.k_2 + \frac{B_{12}}{\beta^2}\ .g_2\left[-\frac{A_1A_3}{\beta^2}k_3 - A_1A_2\beta^2 k_1\right] \\ & + \frac{B_{33}}{\beta^2}.\left[+g_3A_2{}^2 + g_3A_3{}^2\right]k_2 \\ & + \frac{B_{13}}{\beta}\ .\left[4A_1{}^2 - 2g_2(A_1A_2 + A_1A_3) - 4g_2A_1A_2 + 2g_3\left(A_2{}^2 + A_2A_3\right)\right]k_4 \\ & + \frac{B_{22}}{\beta^4}\ .\left[A_1{}^2 - 2g_2A_1A_3 + g_3A_3{}^2\right]k_3 \\ & + \frac{B_{23}}{\beta^3}\ .\left[4A_1{}^2 - 2g_2(A_1A_2 + A_1A_3) - 4g_2A_1A_3 + 2g_3\left(A_2A_3 + A_3{}^2\right)\right]k_5 \\ & + B_{44}.\left(\frac{a}{t}\right)^2.g_4.A_2{}^2k_6 + \frac{B_{55}}{\beta^2}.\left(\frac{a}{t}\right)^2.g_4A_3{}^2k_7\Bigg\} - 2A_1\frac{qa^4}{D_0}k_8\Bigg\} \qquad 44\end{aligned}$$

Note:

$$k_1 = \int_0^1\int_0^1\left(\frac{d^2h}{dR^2}\right)^2 dR\,dQ;\ k_2 = \int_0^1\int_0^1\left(\frac{d^2h}{dRdQ}\right)^2 dR\,dQ\,;\ k_3 = \int_0^1\int_0^1\left(\frac{d^2h}{dQ^2}\right)^2 dR\,dQ$$

$$k_4 = \int_0^1\int_0^1\left(\frac{d^2h}{dR^2}\right)\left(\frac{d^2h}{dRdQ}\right)dR\,dQ;\ k_5 = \int_0^1\int_0^1\left(\frac{d^2h}{dQ^2}\right)\left(\frac{d^2h}{dRdQ}\right)dR\,dQ$$

$$k_6 = \int_0^1\int_0^1\left(\frac{dh}{dR}\right)^2 dR\,dQ;\ k_7 = \int_0^1\int_0^1\left(\frac{dh}{dQ}\right)^2 dR\,dQ;\ k_8 = \int_0^1\int_0^1 h\,dR\,dQ$$

To obtain the quasi equations of equilibrium of forces and quasi compatibility equations, Equation 44 must be differentiated with respect to A1, A2 and A3. That is:

$$\frac{d\Pi}{dA_1} = \frac{d\Pi}{dA_2} = \frac{d\Pi}{dA_3} = 0 \qquad 45$$

$$\frac{d\prod}{dA_1} = L_{11}A_1 - L_{12}A_2 - L_{13}A_3 - \frac{qa^4}{D_0}\ k_8 = 0 \qquad 46$$

$$\frac{d\prod}{dA_2} = L_{12}A_1 - L_{22}A_2 - L_{23}A_3 = 0 \qquad 47$$

$$\frac{d\prod}{dA_3} = L_{13}A_1 - L_{23}A_2 - L_{33}A_3 = 0 \qquad 48$$

Where:

$$L_{11} = B_{11}k_1 + \frac{2\{2B_{33} + B_{12}\}}{\beta^2}k_2 + \frac{B_{22}}{\beta^4}k_3 + 3\frac{B_{13}}{\beta}k_4 + 3\frac{B_{23}}{\beta^3}k_5 \qquad 49$$

$$L_{12} = B_{11}g_2k_1 + \frac{\{2B_{33} + B_{12}\}}{\beta^2}g_2k_2 + 2.25\frac{B_{13}}{\beta}g_2k_4 + 0.75\frac{B_{13}}{\beta}g_2k_5 \qquad 50$$

$$L_{13} = \frac{\{2B_{33} + B_{12}\}}{\beta^2}g_2k_2 + \frac{B_{22}}{\beta^4}g_2k_3 + 0.75\frac{B_{13}}{\beta}g_2k_4 + 2.25\frac{B_{23}}{\beta}g_2k_5 \qquad 51$$

$$L_{12} = B_{11}g_2k_1 + \frac{\{2B_{33} + B_{12}\}}{\beta^2}g_2k_2 + 2.25\frac{B_{13}}{\beta}g_2k_4 + 0.75\frac{B_{13}}{\beta}g_2k_5 \qquad 52$$

$$L_{22} = B_{11}g_3k_1 + \frac{B_{33}}{\beta^2}g_3k_2 + 1.5\frac{B_{13}}{\beta}g_3k_4 + B_{44} \propto^2 g_4k_6 \qquad 53$$

$$L_{23} = \frac{B_{12}}{\beta^2}g_3k_2 + \frac{B_{33}}{\beta^2}g_3k_2 + 0.75\frac{B_{13}}{\beta}g_3k_4 + 0.75\frac{B_{23}}{\beta^3}g_3k_5 + \propto^2 g_4\frac{B_{45}}{\beta}k_8 \qquad 54$$

$$L_{13} = \frac{\{2B_{33} + B_{12}\}}{p^2}g_2k_2 + \frac{B_{22}}{p^4}g_2k_3 + 0.75\frac{B_{13}}{p}g_2k_4 + 2.25\frac{B_{23}}{p}g_2k_5 \qquad 55$$

$$L_{23} = \frac{B_{12}}{p^2}g_3k_2 + \frac{B_{33}}{p^2}g_3k_2 + 0.75\frac{B_{13}}{p}g_3k_4 + 0.75\frac{B_{23}}{p^3}g_3k_5 + \propto^2 g_4\frac{B_{45}}{p}k_8 \qquad 56$$

$$L_{33} = \frac{B_{33}}{p^2}g_3k_2 + \frac{B_{22}}{p^4}g_3k_3 + 1.5\frac{B_{23}}{p^3}g_3k_5 + \propto^2 g_4\frac{B_{55}}{p^2}k_7 \qquad 57$$

Solving Equations 47 and 48 simultaneously gives:

$$A_2 = \left(\frac{L_{12}L_{33} - L_{13}L_{23}}{L_{22}L_{33} - L_{23}L_{23}}\right)A_1 = P_2\ A_1 \qquad 58$$

$$A_3 = \left(\frac{L_{13}L_{22} - L_{12}L_{23}}{L_{22}L_{33} - L_{23}L_{23}}\right)A_1 = P_3\ A_1 \qquad 59$$

Substituting Equations 57 and 58 into Equation 46 gives:

$$A_1 = \frac{qa^4}{D_0} \cdot \frac{k_8}{(L_{11} - L_{12}P_2 - L_{13}P_3)} = \frac{qa^4}{D_0} \cdot k_9 \quad 60$$

g. Development of formulas for thick anisotropic plate analysis

Substituting Equation 60 into Equation 41a and substituting Equation 36 into the resulting equation and simplifying gives:

$$\bar{w} = 12[1 - \mu_{12}\mu_{21}] . k_9 h \quad 61$$

Substituting Equations 41a, 42 and 43 into Equations 8c, 8d, 29, 30, 31, 32 and 33, where appropriate and simplifying gives:

$$\bar{u} = 12[1 - \mu_{12}\mu_{21}] . k_9 . [-S + HP_2] . \frac{\partial h}{\partial R} \quad 62$$

$$\bar{v} = 12[1 - \mu_{12}\mu_{21}] . \frac{[-S + H.P_3]}{\beta} . \frac{\partial h}{\partial Q} . k_9 \quad 63$$

$$\overline{\sigma_R} = 12 . k_9 \left(B_{11} [HP_2 - S] \frac{\partial^2 h}{\partial R^2} + \frac{B_{12}}{\beta^2} [HP_3 - S] \frac{\partial^2 h}{\partial Q^2} + \frac{B_{13}}{\beta} H(P_2 + P_3 - 2S) \frac{\partial^2 h}{\partial R \partial Q} \right) \quad 64$$

$$\overline{\sigma_Q} = 12 . k_9 . \left(B_{21}[HP_2 - S] \frac{\partial^2 h}{\partial R^2} + \frac{B_{22}}{\beta^2} [HP_3 - S] \frac{\partial^2 h}{\partial Q^2} + \frac{B_{23}}{\beta} H(P_2 + P_3) - 2S \frac{\partial^2 h}{\partial R \partial Q} \right) \quad 65$$

$$\overline{\tau_{RQ}} = 12k_9 . \left(B_{31} . [HP_2 - S] \frac{\partial^2 h}{\partial R^2} + \frac{B_{32}}{\beta^2} . [HP_3 - S] \frac{\partial^2 h}{\partial Q^2} + \frac{B_{33}}{\beta} . H(P_2 + P_3 - 2S) \frac{\partial^2 h}{\partial R \partial Q} \right) \quad 66$$

$$\overline{\tau_{RS}} = 12\, k_9 \left(B_{44} . P_2 . \frac{\partial H}{\partial S} \right) . \frac{\partial h}{\partial R} \quad 67$$

$$\overline{\tau_{QS}} = 12 . k_9 . \left(B_{55} . \frac{P_3}{\beta} . \frac{\partial H}{\partial S} \right) . \frac{\partial h}{\partial Q} \quad 68$$

h. Determination of stiffness coefficients

The stiffness coefficients (k) extracted from Equation 44 were solved for four boundary conditions and the unique stiffness coefficient values were obtained for each of the four plate boundary condition and the values are shown on Table 1.

i. Numerical analyses of typical thick anisotropic rectangular plates

The numerical values for typical thick anisotropic rectangular plate in-plane displacements (u and v), out-plane displacement - central deflection (w), in-plane stresses (σ_x, σ_y and τ_{xy}), and out-plane stresses (τ_{xz} and τ_{yz}) were determined for angles fiber orientations of 0^0, 15^0, 30^0, 45^0, 60^0, 75^0 and 90^0 at span to thickness ration (α) of 5, 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 for the twelve boundary conditions considered in this work. The materials were analyzed for in-plane displacement (u,v) at x = 0.5, y = 0.5, z = 0; transverse displacement (w) at x = 0.5, y = 0.5, z = 0); in-plane normal stresses (σ_x, σ_y) at x = 0.5, y = 0.5, z = 0.5 or z = 0.25; in-plane shear stress (τ_{xy}) at x = 0, y = 0, z = 0.5; out-plane shear stress (τ_{xz}) at x = 0, y = 0.5, z = 0 and out-plane shear stress (τ_{yz}) at x = 0.5, y = 0, z = 0. The plate was subjected to uniformly distributed load.

The material properties used are as follows: E_1/E_2 = 25, G_{12}/E_2 = 0.5, G_{13}/E_2 = 0.5, G_{23}/E_2 = 0.2, v_{12} = 0.25. Given data by [11]; E_2/E_1 = 0.52500, G_{12}/E_1 = 0.26293, G_{13}/E_1 = 0.15991, G_{23}/E_1 = 0.26681, μ_{12} = 0.44046, μ_{21} = 0.23124, $(1 - u_{12}u_{21})$ = 0.89815. {Taken from [11]}.

The plate parameters employed here are similar to the one employed by [13]. The following non-dimensionalizations they applied were also used in this work:

$$\bar{w} = w\frac{E_0 t^3}{qa^4}x100,\ \bar{u},\bar{v} = u,v\frac{E_0 t^2}{qa^3},\ (\overline{\sigma_{xx}}, \overline{\sigma_{yy}}, \overline{\tau_{xy}}) = (\frac{\sigma_{x,}\sigma_{y,}\tau_{xy}t^2}{qa^2}),\ (\overline{\tau_{xz}}, \overline{\tau_{yz}}) = (\frac{\tau_{xz,}\tau_{yz}t}{qa}).$$

3. Results and Discussion

a. Total potential energy functional for a thick anisotropic rectangular plate

The total potential energy functional for a thick anisotropic rectangular plate was derived in Equation (35) and is as shown in Equation (69):

equations

The governing equation of equilibrium and two compatibility equations for thick anisotropic rectangular plate which are derived in this study as Equations 38, 39 and 40 are presented in Equations 70, 71 and 72.

$$\int_0^1\int_0^1 \Big\{ B_{11}.\frac{\partial^4 w}{\partial R^4} + \frac{2}{\beta^2}.B_{xy}\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + \frac{B_{22}}{\beta^4}.\frac{\partial^4 w}{\partial Q^4} + 4\frac{B_{13}}{\beta}.\frac{\partial^4 w}{\partial R^3\partial Q} + 4\frac{B_{23}}{\beta^3}.\frac{\partial^4 w}{\partial R\partial Q^3}$$
$$- \frac{g_2 a}{2}[2B_{11} + B_{12}]\frac{\partial^3 \phi_x}{\partial R^3} - \frac{g_2 a}{2\beta^2}.B_{xy}\frac{\partial^3\phi_x}{\partial R\,\partial Q^2} - 3g_2 a.\frac{B_{13}}{\beta}\frac{\partial^3\phi_x}{\partial R^2\partial Q}$$
$$- \frac{g_2 a}{2\beta^3}[B_{12} + 2B_{22}]\frac{\partial^3\phi_y}{\partial Q^3} - \frac{g_2 a}{2\beta}B_{xy}\frac{\partial^3\phi_y}{\partial R^2\partial Q} - 3g_2 a.\frac{B_{23}}{\beta^2}\frac{\partial^3\phi_y}{\partial R\partial Q^2} - g_2 a.B_{13}.\frac{\partial^3\phi_y}{\partial R^3}$$
$$- \frac{g_2 a}{\beta^3}.B_{23}.\frac{\partial^3\phi_x}{\partial Q^3} - \frac{qa^4}{D_0}\Big\} dR\, dQ = 0 \qquad 70$$

$$\mathrm{B}_{11}.\left[-g_2a.\frac{\partial^3 w}{\partial R^3}+g_3a^2.\frac{\partial^2\phi_x}{\partial R^2}\right]+\frac{\mathrm{B}_{12}}{2\beta^2}.\left[-g_2a\beta^2\frac{\partial^3 w}{\partial R^3}-g_2a.\frac{\partial^3 w}{\partial R\,\partial Q^2}+2g_3a^2\beta.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]$$
$$+\frac{B_{13}}{\beta}.\left[-g_2a.\frac{\partial^3 w}{\partial Q\,\partial R^2}-2g_2a\frac{\partial^3 w}{\partial Q\,\partial R^2}+2g_3a^2.\frac{\partial^2\phi_x}{\partial R\partial Q}+g_3a^2.\beta.\frac{\partial^2\phi_y}{\partial R^2}\right]$$
$$+\frac{B_{23}}{\beta^3}.\left[-g_2a.\frac{\partial^3 w}{\partial Q^3}+g_3a^2\beta.\frac{\partial^2\phi_y}{\partial Q^2}\right]+\frac{\mathrm{B}_{33}}{\beta^2}.\left[-g_2a.\frac{\partial^3 w}{\partial R\,\partial Q^2}+g_3a^2.\frac{\partial^2\phi_x}{\partial Q^2}+g_3a^2.\beta\frac{\partial^2\phi_y}{\partial R\partial Q}\right]$$
$$+a^2\mathrm{B}_{44}.\left(\frac{a}{t}\right)^2.g_4.\emptyset_x$$
$$=0 \qquad 71$$

$$\frac{\mathrm{B}_{12}}{2\beta^2}.\left[-g_2\frac{a}{\beta}\frac{\partial^3 w}{\partial Q^3}-g_2a\beta.\frac{\partial^3 w}{\partial R^2\partial Q}+2g_3a^2\beta.\frac{\partial^2\phi_x}{\partial R\partial Q}\right]+\frac{B_{13}}{\beta}.\left[-g_2a.\beta.\frac{\partial^3 w}{\partial R^3}+g_3a^2.\beta.\frac{\partial^2\phi_x}{\partial R^2}\right]$$
$$+\frac{\mathrm{B}_{22}}{\beta^4}.\left[-g_2a\beta.\frac{\partial^3 w}{\partial Q^3}+g_3a^2\beta^2.\frac{\partial^2\phi_y}{\partial Q^2}\right]$$
$$+\frac{B_{23}}{\beta^3}.\left[-g_2a.\beta.\frac{\partial^3 w}{\partial R\,\partial Q^2}-2g_2a\beta.\frac{\partial^3 w}{\partial R\partial Q^2}+g_3a^2\beta.\frac{\partial^2\phi_x}{\partial Q^2}+2g_3a^2\beta^2.\frac{\partial^2\phi_y}{\partial R\partial Q}\right]$$
$$+\frac{\mathrm{B}_{33}}{\beta^2}.\left[-g_2a.\beta\frac{\partial^3 w}{\partial R^2\partial Q}+g_3a^2.\beta\frac{\partial^2\phi_x}{\partial R\partial Q}+g_3a^2.\beta^2.\frac{\partial^2\phi_y}{\partial R^2}\right]+a^2\mathrm{B}_{55}.\left(\frac{a}{t}\right)^2.g_4.\emptyset_y$$
$$=0 \qquad 72$$

c. Stiffness coefficients of the four plate boundary conditions

The stiffness coefficients for the four plate boundary conditions are as listed in Table 1.

Table 1: Stiffness value (k) for rectangular plate

Plate type	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
SSSS	0.23619	0.23592	0.23619	0	0	0.02390	0.02390	0.04000
SCFS	1.50967	0.1823129	0.02872	0.11111	0	0.071889	0.016033	0.0625
CCFS	0.67096	0.040514	0.006047	0	0	0.0159753	0.003376	0.027778
SCFC	4.16000	0.8816327	0.217985	0.5625	0	0.198095	0.077531	0.09000

Table 2: Numerical values of displacements and stresses for SSSS thick anisotropic rectangular plate for 0^0 @ α = 5 to 100, β = 1

α	$\overline{\mathbf{w}}$	$\overline{\mathbf{u}}$	$\overline{\mathbf{v}}$	$\overline{\sigma_{xx}}$	$\overline{\sigma_{yy}}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	0.0180	-0.32005	-0.60615	0.980772	0.0364545	0.0592765	0.59966	0.06820

10	0.01005	-1.15660	-1.51589	0.881023	0.0259871	0.0427599	0.67721	0.05531
20	0.00775	-4.4924	-4.87597	0.853597	0.0222705	0.0374734	0.70067	0.05011
30	0.00731	-10.0509	-10.4394	0.848404	0.0215162	0.0364274	0.70524	0.04903
40	0.00715	-17.8329	-18.2231	0.846577	0.0212470	0.0360560	0.70686	0.04864
50	0.00708	-27.8382	-28.2292	0.845730	0.0211214	0.0358832	0.70761	0.04846
60	0.00704	-40.0669	-40.4583	0.845270	0.0210529	0.0357890	0.70802	0.04836
70	0.00701	-54.5190	-54.9107	0.844992	0.0210115	0.0357321	0.70827	0.04830
80	0.0070	-71.1945	-71.5863	0.844811	0.0209846	0.0356952	0.70843	0.04826
90	0.00699	-90.0934	-90.4853	0.844688	0.0209662	0.0356699	0.70854	0.04823
100	0.00698	-111.216	-111.608	0.844599	0.0209530	0.0356517	0.70862	0.04821

Table 3: Numerical values of displacements and stresses for SCFS thick anisotropic rectangular plate for 0^0 @ α = 5 to 100, β = 1

α	$\bar{w}$	$\bar{u}$	$\bar{v}$	$\overline{\sigma_{xx}}$	$\overline{\sigma_{yy}}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	0.01142	-0.04699	-0.17713	0.56898	0.02061	-0.01066	0.14369	0.00346
10	0.00464	-0.13268	-0.29131	0.40058	0.01014	-0.00518	0.14867	0.00336
20	0.00287	-0.47306	-0.72846	0.35664	0.00740	-0.00375	0.14997	0.00333
30	0.00253	-1.04010	-1.45500	0.34842	0.00689	-0.00348	0.15021	0.00332
40	0.00242	-1.83393	-2.47189	0.34554	0.00671	-0.00338	0.15030	0.00332
50	0.00236	-2.85455	-3.77925	0.34420	0.00662	-0.00334	0.15034	0.00332
60	0.00233	-4.10198	-5.37710	0.34348	0.00658	-0.00332	0.15036	0.00332

70	0.00232	-5.57622	-7.26545	0.34304	0.00655	-0.00330	0.15037	0.00332
80	0.00231	-7.27725	-9.44431	0.34275	0.00653	-0.00329	0.15038	0.00332
90	0.00230	-9.20509	-11.91369	0.34256	0.00652	-0.00329	0.15039	0.00332
100	0.00229	-11.35974	-14.67358	0.34242	0.00651	-0.00328	0.15039	0.00332

Table 4: Numerical values of displacements and stresses for CCFS thick anisotropic rectangular plate for 0^0 @ $\alpha = 5$ to 100, $\beta = 1$

α	$\bar{w}$	$\bar{u}$	$\bar{v}$	$\overline{\sigma_{xx}}$	$\overline{\sigma_{yy}}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	0.00159	-0.00652	-0.03598	0.07930	0.00404	-0.00159	0.01999	0.00107
10	0.00066	-0.01882	-0.06116	0.05696	0.00195	-0.00080	0.02114	0.00088
20	0.00041	-0.06750	-0.15426	0.05097	0.00138	-0.00058	0.02145	0.00081
30	0.00036	-0.14858	-0.30860	0.04984	0.00127	-0.00054	0.02151	0.00080
40	0.00035	-0.26207	-0.52455	0.04945	0.00123	-0.00052	0.02153	0.00080
50	0.00034	-0.40799	-0.80217	0.04926	0.00122	-0.00052	0.02154	0.00080
60	0.00033	-0.58634	-1.14147	0.04916	0.00121	-0.00051	0.02154	0.00079
70	0.00033	-0.79711	-1.54246	0.04910	0.00120	-0.00051	0.02154	0.00079
80	0.00033	-1.04031	-2.00513	0.04906	0.00120	-0.00051	0.02155	0.00079
90	0.00033	-1.31594	-2.52949	0.04904	0.00120	-0.00051	0.02155	0.00079
100	0.00033	-1.62399	-3.11555	0.04902	0.00119	-0.00051	0.02155	0.00079

Table 5: Numerical values of displacements and stresses for SCFC thick anisotropic rectangular plate for 0^0 @ α = 5 to 100, β = 1.5

α	$\bar{w}$	$\bar{u}$	$\bar{v}$	$\overline{\sigma_{xx}}$	$\overline{\sigma_{yy}}$	$\overline{\tau_{xy}}$	$\overline{\tau_{xz}}$	$\overline{\tau_{yz}}$
5	**0.00020**	**-0.00257**	**-0.00509**	**-0.00045**	**-0.00003**	**0.00063**	**0.00786**	**0.00007**
10	**0.00008**	**-0.00722**	**-0.00827**	**-0.00031**	**-0.00001**	**0.00030**	**0.00808**	**0.00008**
20	**0.00005**	**-0.02569**	**-0.02060**	**-0.00028**	**-0.00001**	**0.00022**	**0.00814**	**0.00009**
30	**0.00004**	**-0.05646**	**-0.04112**	**-0.00027**	**-0.00001**	**0.00020**	**0.00815**	**0.00009**
40	**0.00004**	**-0.09955**	**-0.06984**	**-0.00027**	**-0.00001**	**0.00020**	**0.00815**	**0.00009**
50	**0.00004**	**-0.15494**	**-0.10677**	**-0.00027**	**-0.00001**	**0.00019**	**0.00815**	**0.00009**
60	**0.00004**	**-0.22264**	**-0.15191**	**-0.00027**	**-0.00001**	**0.00019**	**0.00816**	**0.00009**
70	**0.00004**	**-0.30266**	**-0.20525**	**-0.00027**	**-0.00001**	**0.00019**	**0.00816**	**0.00009**
80	**0.00004**	**-0.39498**	**-0.26680**	**-0.00027**	**-0.00001**	**0.00019**	**0.00816**	**0.00009**
90	**0.00004**	**-0.49961**	**-0.33655**	**-0.00027**	**-0.00001**	**0.00019**	**0.00816**	**0.00009**
100	**0.00004**	**-0.61655**	**-0.41451**	**-0.00026**	**-0.00001**	**0.00019**	**0.00816**	**0.00009**

d. Discussion of results

i. Total potential energy functional

The total potential energy functional for the thick anisotropic rectangular plate derived in this work can be used to analyze rectangular thick anisotropic plate of any boundary condition and it was used here to solve for rectangular plate subjected under pure bending loading. Although, it can also solve buckling loading and vibration loading when the external work is substituted appropriately. It is presented here in the expanded form to accommodate the exact displacement functions unlike that of other works like [11, 12, 13] that assumed their displacement functions and had no need for such expansion. Hence it is similar when compare with other works in a minimized form but exhibit some level of differences when compared in the expanded form. The above statements can be relied on to confirm the efficiency of this Equation (69) for the analysis of thick anisotropic rectangular plate. The equation is a combination of differential of central deflection (w), in-plane rotational displacement (ϕ_x) on x-axis and in-plane rotational displacement (ϕ_y) on y-axis. Previous work on thick anisotropic rectangular plate did not care to separate the two rotational in-plane displacements (ϕ_x, ϕ_y) because they assumed their displacement functions and has no need for further expansion.

ii. Governing equation and compatibility equations

The total potential energy was minimized with displacements to obtain the governing equation of equilibrium and the two compatibility equations of thick anisotropic plate. The governing equation obtained in Equation 143 is similar to those obtained by [11, 12], etc, but their various works were able to obtained one compatibility equation in addition to their governing equation unlike the present work that obtained two compatibility Equations (71) and (72) The governing equation comprises the external work (pure bending loading), differentials of out-plane displacement (W), differential of in-plane rotational displacement in x-axis and differential of in-plane rotational displacement in y-axis. The compatibility equations contain the differentials of out-plane displacement (W), differential of in-plane rotational displacement in x-axis, differential of in-plane rotational displacement in y-axis, whole in-plane rotational displacement in x-axis and whole in-plane rotational displacement in y-axis.

iii. Polynomial stiffness values (k) of the rectangular plates

The polynomial stiffness values (k) of the rectangular plate for the four boundary conditions considered here were the product of closed form integral of the displacement functions. The Equations and the values obtained are similar to those obtained by [13]. Table 1 showed that the polynomial stiffness values obtained for the various rectangular plate boundary conditions (SSSS and CCFC) yielded the same value, zero for, (k_4 and k_5). SCFS and SCFC rectangular plate boundary conditions yielded 0.1111 and 0.5625 for its (k_4) stiffness values. Thus, for the four boundary conditions considered, the stiffness values of k_5, yielded zero in all boundary conditions while k_4 yielded values for only two boundary conditions as stated above. These yield of the same values for (k_5) and most (k_4) confirmed the similarities in its Equations.

iv. SSSS plate at angle fiber orientation of 0^0

From Table 2, it is observed that out-plane displacement values $\overline{w}$ decreases as the thickness of the plate decreases. The decrease was very high at the thick plate zone (α = 5 to 10) but becomes very small at the thing plate zone (α = 50 to 100). This is a confirmation that the out-plane displacement act more on thick plate zone than thin plate zone in rectangular thick anisotropic plate. The in-plane displacements, $\bar{u}$ and $\bar{v}$, also decrease as the thickness of the plate decreases and become more noticeable at the thin plate section (α = 50 to 100). This shows that the in-plane displacements have a minimal effect on SSSS thick anisotropic plate.

The in-plane stresses, $\overline{\sigma_{xx}}, \overline{\sigma_{yy}}$ and $\overline{\tau_{xy}}$, decrease as the plate decreases in thickness. A close look will reveal a sharp decrease at thick and moderately thick plate section (α = 5 to 20) while at the thin plate section (α = 50 to 100), they decreased lightly. The out-plane stress, $\overline{\tau_{xz}}$, increases as the plate thickness decreases while the out-plane stress, $\overline{\tau_{yz}}$, decreases as the plate thickness decreases. This divergence in the progressive order of the plate values can be explained from the fact that anisotropic plates are plates with different resistance to mechanical actions in different directions. That is, they possess different properties in different directions.

v. SCFS plate at angle fiber orientation of 0^0

Table 3, shows that the values of out-plane displacement, ($\overline{w}$), increases as the thickness of the plate decreases while the values of the in-plane displacement, ($\bar{u}$ and $\bar{v}$), decrease as the thickness of the plate decreases and becomes more noticeable at the thin plate section, (α = 50 to 100). Also, the values of the in-plane stresses, ($\overline{\sigma_{xx}}$ and $\overline{\sigma_{yy}}$), and out-plane stresses, ($\overline{\tau_{xz}}$ and $\overline{\tau_{yz}}$), increase as the thickness of the plate decreases while the values of the in-plane stress, ($\overline{\tau_{xy}}$), decreases as the plate thickness decreases. This increase or decrease in stresses as the plate thickness decreases are very obvious at the thick plate section, (α = 5 to 10), but gradually decreases as the thickness of the plate decreases.

vi. CCFS plate at angle fiber orientation of 0^0

Table 4, shows that the values of out-plane displacement, ($\overline{w}$), and in-plane displacements, ($\overline{u}$ and $\overline{v}$), decrease as the thickness of the plate decreases. The decrease is more obvious at the thick plate section, (α = 5 to 10). Hence, displacements have more effect on thick plate than on thin plate. Also, the values of the in-plane stresses, ($\overline{\sigma_{xx}}$ and $\overline{\sigma_{yy}}$), and out-plane stress, ($\overline{\sigma_{yz}}$), decrease as the thickness of the plate decreases while the values of the in-plane stresses, ($\overline{\tau_{xy}}$), out-plane stress, ($\overline{\tau_{xz}}$), increase as the plate thickness decreases. This increase or decrease in stresses as the plate thickness decreases are very obvious at the thick plate section, (α = 5 to 10), but gradually decrease as the thickness of the plate decreases. Hence, the impact from stresses are felt more on thick plate than thin plate.

vii. SCFC plate at angle fiber orientation of 0^0

From Table 5, it is observed that the values of displacements, ($\overline{w}$, $\overline{u}$, $\overline{v}$), and the in-plane stress, ($\overline{\tau_{xy}}$), decrease as the thickness of the plate decreases while the in-plane stresses, ($\overline{\sigma_{xx}}$, $\overline{\sigma_{yy}}$), and the out-plane stresses, ($\overline{\tau_{yz}}$, $\overline{\tau_{xz}}$), increase as the thickness of the plate decreases. However, the out-plane displacement, ($\overline{w}$), and stresses, ($\overline{\sigma_{yy}}$, $\overline{\sigma_{yy}}$, $\overline{\tau_{xy}}$, $\overline{\tau_{xz}}$, $\overline{\tau_{yz}}$), showed a very unique sequence between preceding values starting from moderate thick plate zone, (α = 20 to 40), the increase or decrease between two successive preceding values, becomes very small and even diminishes to infinitesimal difference at the thin plate zone (α = 50 to 100). The decrease or the increase of the displacements and stresses have a wider margin between the preceding values at the thick plate zone, (α = 5 to 10), but gradually diminishes towards moderate thick and thin plate zone, (α = 20 to 100). The values obtained for displacements, ($\overline{w}$, $\overline{u}$, $\overline{v}$), and stresses, ($\overline{\sigma_{xx}}$, $\overline{\sigma_{yy}}$, $\overline{\tau_{xy}}$, $\overline{\tau_{xz}}$, $\overline{\tau_{yz}}$), at the minimum span to thickness ratio of (5) are, (0.00020, -0.00257, -0.00509) and (-0.00045, -0.00003, 0.00063, 0.00786, 0.00007) while at the maximum span to thickness ratio of (100), it yield, (0.00004, -0.61655, -0.41451) and (-0.00026, -0.00001 -0.00019, 0.00816, 0.00009).

4. Numerical problems comparisons

In this section, the present study results were compared with the existing literature results obtained by some other scholars by calculating the percentage difference between the present study and the existing literature results by the scholar.

The percentage difference is calculated as follows:

$$\% \text{ difference} = \left(\frac{\text{present study value} - \text{comparing author value}}{\text{present study value}}\right) X\ 100\%$$

Table 6: Comparison of results of non-dimensional deflection, w, from present study with that of [13] or rectangular anisotropic plate at 0^0 angle fiber orientation. And aspect ratio of 1 (b/a = 1)

a/t	Author	$\overline{w}$
10	Atashipour et.al.(2017)	0.9520
	Present study	1.0051
	%difference	**5.28%**

20	Atashipour et.al.(2017)	0.7262
	Present study	0.7749
	%difference	**6.28%**
100	Atashipour et.al.(2017)	0.6528
	Present study	0.6981
	%difference	**6.49%**

From Table 6, it is observed that, present study deflections (w), have considerable similarity with the deflections obtained by [13] in their single layer orthotropic square plate solutions with percentage differences of 5.28%, 6.28% and 6.49% at span to thickness ratios (a/t) of 10, 20 and 100 and aspect ratio of, 1, respectively. The following factors may have contributed to the mild differences between the present study results and that of [13];

i. [13] applied Levy type solution in Fourier differential quadrature while present study used Ritz energy method through exact approach.

ii. [13] employed modification factor of 5/6 while present study does not require modification factor.

iii. [13] adopted first order shear deformation theory while present study adopted third order shear deformation theory.

iv. [13] employed assumed displacement functions while present study used exact displacement functions.

Table 7: Comparison of present study non-dimensional out-plane displacement ($\overline{w}$) of simply supported orthotropic rectangular plate under uniformly distributed transverse load with that of [11]

Plate dimensional parameters		$\overline{w}$, at x = a/2, y = b/2, z = t/2		
		Present study	Shimpi & Patel (2006)	% difference
b/a	a/t			
2.0	20.0	22034	21542	2.22%
	10.0	1451.90	1408.5	2.98%
	7.14286	402.76	387.23	3.86%
1.0	20.0	10,382	10443	0.59%
	10.0	678.57	688.57	1.47%

	7.14286	186.05	191.07	2.70%
0.5	20.0	2054.06	2048.7	0.26%
	10.0	139.46	139.08	0.27%
	7.14286	39.86	39.79	0.18%

Nondimensional out-plane displacement ($\overline{w}$) of simply supported orthotropic rectangular plate under uniformly distributed transverse load was analyzed through exact approach by applying Ritz energy method using polynomial shear deformation functions. The results were presented in Table 7. The table also present results obtained by [11] using refined plate theory. This refined plate theory results by [11] were used as basis for comparison of the results obtained in the present study.

The present study exact approach results converged well with the results obtained by [11] with maximum percentage difference being 3.86% for rectangular plate of width/length ratio (b/a = 2), span/thickness ratios (a/t = 7.14286). The minimum percentage difference (0.18%) occurred in the rectangular plate of width/length ratio (b/a = 0.5), span/thickness ratios (a/t = 7.14286). The average percentage difference for the three plates with three span to thickness ratios each is 1.614%. This serves as a complimentary result to the earlier submission that the obtained orthotropic thick plate displacement shows good accuracy with that of [11]. From Table 7, it can be stated that the lesser the aspect ratio the better the out-plane displacement results.

Table 8: Comparison of present study non-dimensional in-plane stress ($\overline{\sigma_{xx}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load with that of [11].

Plate dimensional parameters		$\overline{\sigma_{xx}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Shimpi & Patel (2006)	% difference
b/a	a/t			
2.0	20.0	262.05	262.67	0.24%
	10.0	65.89	65.975	0.13%
	7.14286	33.863	33.862	0.003%
1.0	20.0	143.425	144.31	0.62%
	10.0	35.59	36.021	1.20%
	7.14286	18.002	18.346	1.91%

Table 8 shows the comparison of present study results with that of [11] for non-dimensional in-plane stress ($\overline{\sigma_{xx}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load. The results shows high level of convergence with very low percentage difference for rectangular plate with aspect ratio 2 (b/a = 2) and span to thickness ratios (a/t = 20, 10 and 7.14286). The percentage differences obtained for these aspect ratio and span to thickness ratios are; 0.24%, 0.13% and 0.003%. Square plate also showed high level of convergence with low percentage differences of 0.62, 1.20% and 1.91% for span to thickness ratios (a/t = 20, 10 and 7.14286). From Table 8, it is observed that for in-plane stress ($\overline{\sigma_{xx}}$), the higher the aspect ratio (b/a) the better the results.

Table 9: Comparison of present study non-dimensional in-plane stress ($\overline{\sigma_{yy}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load with that of [11].

Plate dimensional parameters		$\overline{\sigma_{yy}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Shimpi & Patel (2006)	% difference
b/a	a/t			
1.0	20.0	88.24	87.08	1.31%
	10.0	21.997	22.21	0.97%
	7.14286	11.178	11.615	3.91%

Table 9 presented the values of non-dimensional in-plane stress ($\overline{\sigma_{yy}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load as obtained by present study and [11]. The present study results were compared with the results obtained by [11] and it shows lower percentage differences for the square plate at a/t = 20, 10 and 7.14286 with percentage differences of 1.32%, 0.97% and 3.91%. It is observed from Table 9 that, the present study results converges very well with those from [11] when solving in-plane stress ($\overline{\sigma_{yy}}$) for square plate.

Table 10: Comparison of present study non-dimensional out-plane stress ($\overline{\tau_{xz}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load with those from [15, 12, 16 & 11].

Plate dimensional parameters		$\overline{\tau_{xz}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Srinivas et al. (1970)	% difference
b/a	a/t			
2.0	20.0	13.635	14.048	3.03%
	10.0	6.8082	6.9266	1.74%
	7.14286	4.8544	4.8782	0.49%

Plate dimensional parameters		$\overline{\tau_{xz}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Reddy (1984)	% difference
b/a	a/t			
2.0	20.0	13.635	13.98	2.53%
	10.0	6.8082	6.958	2.20%
	7.14286	4.8544	4.944	1.85%
Plate dimensional parameters		$\overline{\tau_{xz}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Reissner (1945)	% difference
b/a	a/t			
2.0	20.0	13.635	14.114	3.51%
	10.0	6.8082	7.0611	3.71%
	7.14286	4.8544	5.0445	3.92%
Plate dimensional parameters		$\overline{\tau_{xz}}$, at x = a/2, y = b/2, z = t/2		
		Present study	Shimpi & Patel (2006)	% difference
b/a	a/t			
2.0	20.0	13.635	14.03	2.89%
	10.0	6.8082	6.78	0.41%
	7.14286	4.8544	4.70	3.18%

Table 10, shows the results comparison for out-plane stress ($\overline{\tau_{xz}}$) of simply-supported orthotropic rectangular plate under uniformly distributed transverse load as obtained by various authors with various theories. [15] used exact theory, [12] used higher order shear deformation plate theory, [16] used first order shear deformation plate theory while [11] used refined plate theory. It is interesting to note that all the methods presented in Table 10 in exception to present study solution are either moments or stress based approach or both. Also, the shear deformation function they applied were different from the shear deformation function of the present study. Their shear deformation functions are stated herein: [15] and [11]used $f(z) = \left[\frac{1}{4}\left(\frac{z}{t}\right) - \frac{5}{3}\left(\frac{z}{t}\right)^3\right]$, Reddy (1984) applied $f_1(z) = -C_0 z - C_3 z^3$ and $f_2(z) = -C_1 z - C_3 z^3$, Reissner (1945) used $f(z) = \frac{z}{2}\left[\frac{h^2}{4} - \frac{z^2}{3}\right]$ while present study used $f(z) = z\left(1 - \frac{4}{3}\left[\frac{z}{t}\right]^2\right)$ as shear deformation function. All the solutions yielded close values for stress $\overline{\tau_{xz}}$ even with different shear deformation functions. The work by [15] presented the lowest percentage differences when compared with the present study with percentage differences of, 3.03%, 1.74% and 0.49% for b/a =2, a/t = 20, 10 and 7.14286 while the solution from [16] had highest percentage difference on comparing with the present study with differences of 3.51%, 3.71% and 3.92% for the same geometric properties. However, other solutions gave values that are still below 3.2% percentage difference which also is a prove of good agreement between them and present study. Reddy percentage differences with present study are 2.53%, 2.20% and 1.85% while [11] results yielded 2.89%, 0.41% and 3.18% respectively. The rectangular plate considered has the following geometric parameters; b/a = 2, a/t = 20, 10 and 7.14286 for all the authors considered on Table 10.

5. Conclusion and recommendations

a. Conclusion

The study presents a solution for the analysis of thick rectangular anisotropic plates based on third order shear deformation theory and assumptions. Ritz energy method was employed for the analysis. The solution derived the general orthogonal polynomial displacement functions for a rectangular plate from the governing equation of equilibrium and compatibility equations of a rectangular thick anisotropic plate based on third order shear deformation theory. The shear deformation function used was determine from the first principle. Deflection at the center of the anisotropic rectangular plate was determined at “0^0” angle fiber orientation, various span to thickness ratios, α (5, 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100) and for all the four boundary conditions considered in this work, namely: SSSS, SCFS, CCFS and SCFC. In-plane displacements (u and v), in-plane stresses (σ_x, σ_y and τ_{xy}) and out-plane stresses (τ_{xz} and τ_{yz}) were also determined for the same angles of orientation of fibers, span-depth-ratios and boundary conditions as applied to central deflection. Finally, a functional excel worksheet program was developed for easy analysis of thick anisotropic plates.

The total potential energy functional developed for the rectangular thick anisotropic plate using third order shear deformation theory, the formulated governing equation of equilibrium and the compatibility equations of anisotropic plate, the stiffness coefficients, the orthogonal polynomial shear deformation and the exact displacement functions developed in this work can be used to provide satisfactory solution to anisotropic thick rectangular plate problems.

b. Recommendations

This research work used third order shear deformation theory to analyze thick anisotropic rectangular plate through exact approach with twelve boundary conditions. Thus; it is recommended that:

i. The method shall be used when analyzing thick anisotropic rectangular plate due to it suitability and usability.

ii. Further studies shall use exact approach in third order shear deformation theory in solving other related anisotropic plate problems like thin laminated anisotropic plate, thin layered anisotropic plate and laminated functionally graded anisotropic plate.

iii. Further studies shall use exact approach in third order shear deformation theory in other methods than the Ritz energy method, such as in the Galerkin method, the Kantorovich method, the Trefftz method and the method of least squares.

iv. Further studies shall use exact solution in third order shear deformation theory to analyze thick anisotropic non-rectangular plate problems.

v. Further studies shall use exact approach in third order shear deformation theory to solve thick anisotropic plate with different laminars.

References

[1] Vasiliev, V. V., and Morozov, E. (2013). Advanced mechanics of composite materials and structural elements (third Edition). Elsevier: London

[2] Ibearugbulem, O.M., Ezeh, J.C., Ozioko, H.O. and Anya, U.C. (2020). Novel Formulas for Displacements and Stresses of Thick Anisotropic Rectangular Plate. Saudi Journal of Civil Engineering, 4(9), 137-149. DOI: 10.36348/sjce.2020.v04i09.002.

[3] Reddy, J.N., and Arciniega, R. A. (2004). Shear deformation plate and shell theories: from Stavsky to present. Journal of Mechanics of Advanced Material and Structures, 11(6), 535–582. https://doi.org/10.1080/15376490490452777

[4] Ezeh, J. C., Ibearugbulem, O. M. Anya, U. C. and Ozioko, H. O. (2020). Analysis of SSSS and CCCC Thick Anisotropic Rectangular Plate Using Exact Displacement Function. Saudi Journal of Civil Engineering, 4(9), 150-160. DOI: 10.36348/sjce.2020.v04i09.003.

[5] Lekhnitskiy, S.G. (1968). Anisotropic plates (English translation of the original Russian work). Gordon and Breach: New York.

[6] Caliri Jr, M. F., Ferreira A.J.M., and Tita V. (2016). A review on plate and shell theories for Laminated and sandwich structures highlighting the finite element method. Journal of Composite Structures, 156(1), 63 – 77. https://www.elsevier.com/locate/compstruct

[7] Rakocevic, M., Popovic, S., and Ivanisevic, N. (2017). A computational method for laminated composite plates based on layerwise theory. Journal of Composite Part B Engineering, 122(2), 202 – 218. https://doi.org/10.1016/j.compositesb.2017.03.044

[8] Sahoo, R., and Singh, B.N. (2013). A new shear deformation theory for the static analysis of laminated composite and sandwich plates. International Journal of Mechanical Science, 75(8), 324 – 336. https://doi.org/10.1016/j.ijmecsci.2013.08.002

[9] Ozioko, H.O., Ibearugbulem, O.M., Ezeh, J.C. and Anya, U.C. (2019). Algorithm for Exact Solution of Thick Anisotropic Plates. Scholarly Journal of Applied Science Research, 2(4), 11-25. https://www.innovationinfo.org

[10] Ozioko, H.O., Ibearugbulem, O.M. and Anya, U.C. (2020). Determination of Displacement Functions of Thick Anisotropic Rectangular Plate by Direct Integration of Governing Equation. International Journal of Latest Technology in Engineering, Management & Applied Science, 9 (1), 67 – 71. https://www.ijltemas.in

[11] Shimpi, R. P., and Patel, H. G. (2006). A two variable refined plate theory for orthotropic plate analysis. International Journal of Solids and Structures, 43(22-23), 6783-6799. https://doi.org/10.1016/j.ijsolstr.2006.02.007

[12] Reddy, J. N. (1984). A simple higher-order theory for laminated composite plate. Journal of Applied Mechanics, 51(1), 745 – 752. https://doi.org/10.1115/1.3167719

[13] Atashipour, S.R., Girhammar, U.A., and Al-Emrani, M. (2017). Exact Lévy-type solutions for bending of thick laminated orthotropic plates based on 3-D elasticity and shear deformation theories. Journal of Composite Structures, 163(5), 129 – 151. https://doi.org/10.1016/j.compstruct.2016.12.026

[14] Ibearugbulem, O.M., Ezeh, J.C., and Ettu, L.O. (2014). Energy Methods in Theory of Rectangular Plates. LIU House of excellence ventures Publishing: 173, Tetlow Road, Owerri, Imo State, Nigeria.

[15] Srinivas, S., JogaRao, C.V., and Rao, A.K. (1970). An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates. Journal of Sound and Vibration, 12(4), 187– 199. https://doi.org/10.1016/0022-460X(70)90089-1

[16] Reissner, E. (1945). The effect of transverse shear deformation on the bending of elastic plates. Journal of Applied Mechanics, 12(2), 69–77.